A The Quadratic Solver

Given a quadratic solver, we would like to formulate upper bounds on the computed root errors and the residual errors.

A.1 The Root Error

We use the quadratic formula to compute the roots, which involves `square_root` and `divide` operations. Because of that, we cannot use Theorem 4.1 to derive the error bounds. Without losing the generality, we assume that the quadratic equation is $at^2+bt+c=0$, where $a>0$. To avoid cancellation catastrophe [Goldberg 1991] caused by `subtract`, we use the following formula to compute the two roots as proposed in [Goldberg 1991]:

$$\begin{cases} t_0 = \frac{2c}{-b+\sqrt{\Delta}}, \\ t_1 = \frac{-b+\sqrt{\Delta}}{2a}, \end{cases} \text{if } b \le 0; \qquad \begin{cases} t_0 = \frac{-b-\sqrt{\Delta}}{2a}, \\ t_1 = \frac{2c}{-b-\sqrt{\Delta}}, \end{cases} \text{if } b > 0; \tag{4}$$

in which $\Delta = \texttt{Clamp}(b^2-4ac, 0, +\infty)$ is a revised discriminant. The `Clamp` step prevents roots from being eliminated due to the rounding errors. If no exact root exists in $[0,1]$ indeed, the above formula may introduce a false root. Fortunately, it affects the computational cost, but not the safety of collision detection.

Root t_0 when $b \le 0$. The root t_0 exists in $(0,1)$ only when $c \ge 0$ and $b^2 \ge 4ac$. Using Theorem 4.1, we know the error associated with Δ is bounded by: $(b^2+4ac)((1+\epsilon)^2-1) \le 4b^2\epsilon + b^2O(\epsilon^2)$. This bound is still valid after `Clamp`, which makes the error even smaller. Since $\sqrt{|A \pm E|} \le \sqrt{A} + \sqrt{E}$ for any two positive numbers A an E, the error associated with $\sqrt{\Delta}$ is bounded by: $(\sqrt{\Delta}+2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon))\epsilon + 2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon) \le 2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon)$. We then derive an error bound for $-b+\sqrt{\Delta}$ as: $(-b+\sqrt{\Delta}+2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon))\epsilon + 2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon) \le 2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon)$. Let e be the error associated with $-b+\sqrt{\Delta}$. Using Taylor expansion, we get:

$$\left|t_0 - \frac{2c}{-b+\sqrt{\Delta}+e}\right| \le t_0 \left|1 - \frac{1}{1+\frac{e}{-b+\sqrt{\Delta}}}\right| \le \left|\frac{e}{-b+\sqrt{\Delta}}\right| + O(\epsilon) \le 2\epsilon^{\frac{1}{2}} + O(\epsilon). \tag{5}$$

So the root error is bounded by: $|\hat{t}_0 - t_0| \le \left(t_0 + 2\epsilon^{\frac{1}{2}} + O(\epsilon)\right)\epsilon + (2\epsilon^{\frac{1}{2}} + O(\epsilon)) \le 2\epsilon^{\frac{1}{2}} + O(\epsilon)$.

Root t_1 when $b \le 0$. If $b \le 0$ and $c > 0$, then the error associated with $-b+\sqrt{\Delta}$ should still be bounded by $2|b|\epsilon^{\frac{1}{2}} + |b|O(\epsilon)$, as shown previously. Let e be the error associated with $-b+\sqrt{\Delta}$. We have:

$$\left|t_1 - \frac{-b+\sqrt{\Delta}+e}{2a}\right| = t_1 \left|\frac{e}{-b+\sqrt{\Delta}}\right| \le \frac{|e|}{|b|} \le 2\epsilon^{\frac{1}{2}} + O(\epsilon). \tag{6}$$

Therefore, the root error $|\hat{t}_1 - t_1|$ is still bounded by $2\epsilon^{\frac{1}{2}} + O(\epsilon)$.

If $b \le 0$ and $c \le 0$, both b^2 and $-4ac$ are non-negative. In that case, the error associated with Δ is bounded by: $((b^2-4ac)(1+\epsilon)^2-1) \le 2\epsilon\Delta + O(\epsilon^2)\Delta$. We can then formulate the error bound on $\sqrt{\Delta}$ as: $(\sqrt{\Delta}+\sqrt{2\Delta}\epsilon^{\frac{1}{2}}+\sqrt{\Delta}O(\epsilon))\epsilon+\sqrt{2\Delta}\epsilon^{\frac{1}{2}}+\sqrt{\Delta}O(\epsilon) \le \sqrt{\Delta}(\sqrt{2}\epsilon^{\frac{1}{2}}+O(\epsilon))$, and the error bound on $-b+\sqrt{\Delta}$ as: $(-b+\sqrt{\Delta})\epsilon + \sqrt{\Delta}(\sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon))(1+\epsilon) \le (-b+\sqrt{\Delta})(\sqrt{2}\epsilon^{\frac{1}{2}}+O(\epsilon))$. Let e be the error associated with $-b+\sqrt{\Delta}$. We have:

$$\left|t_1 - \frac{-b+\sqrt{\Delta}+e}{2a}\right| = t_1 \left|\frac{e}{-b+\sqrt{\Delta}}\right| \le \sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon). \tag{7}$$

Therefore, $|\hat{t}_1 - t_1| \le (t_1 + \sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon))\epsilon + \sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon) \le 2\epsilon^{\frac{1}{2}}$.

Root t_0 when $b > 0$. Since $a > 0$, t_0 is negative and invalid.

Root t_1 when $b > 0$. If $b > 0$, we must have $c \le 0$ to make $t_1 \ge 0$. If so, the error bound of $\sqrt{\Delta}$ is: $\sqrt{\Delta}(\sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon))$, and the error bound of $b+\sqrt{\Delta}$ is: $(b+\sqrt{\Delta})(\sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon))$, as shown previously. Let e be the actual error associated with $-b-\sqrt{\Delta}$. Using Taylor expansion, we get:

$$\left|t_1 - \frac{2c}{-b-\sqrt{\Delta}+e}\right| \le t_1 \left|1 - \frac{1}{1+\frac{e}{-b-\sqrt{\Delta}}}\right| \le \left|\frac{e}{-b-\sqrt{\Delta}}\right| + O(\epsilon) \le \sqrt{2}\epsilon^{\frac{1}{2}} + O(\epsilon). \tag{8}$$

So we have $|\hat{t}_1 - t_1| \le 2\epsilon^{\frac{1}{2}}$.

Summary. In summary, $2\epsilon^{\frac{1}{2}} + O(\epsilon)$ is an upper bound on the root error in all of the four cases. This is valid even after `Clamp`, which makes the error even smaller.

A.2 The Residual Error

While the exact root t satisfies $at^2+bt+c=0$, the computed root $\hat{t}$ may not. So we would like to formulate an upper bound on $\left|a\hat{t}^2 + b\hat{t} + c\right|$. Again, the roots can be computed in three ways.

Root t_0 when $b \le 0$. Let $\bar{t}_0 = \frac{2c}{-b+\sqrt{\hat{\Delta}}}$ be the root when only the discriminant error exists. When $c \ne 0$, $t_0 \ne 0$ and $\bar{t}_0 \ne 0$. We have:

$$\begin{aligned} \left|a + \frac{b}{\bar{t}_0} + \frac{c}{\bar{t}_0^2}\right| &= \left|a + \frac{b}{\bar{t}_0} + \frac{c}{\bar{t}_0^2} - a - \frac{b}{t_0} - \frac{c}{t_0^2}\right| \\ \le \left|\frac{b(-b+\sqrt{\hat{\Delta}})}{2c} + \frac{(-b+\sqrt{\hat{\Delta}})^2}{4c} - \frac{b(-b+\sqrt{\Delta})}{2c} - \frac{(-b+\sqrt{\Delta})^2}{4c}\right| &= \left|\frac{\hat{\Delta}-\Delta}{4c}\right|. \end{aligned} \tag{9}$$

Since $\left|\hat{\Delta} - \Delta\right| \le (b^2+4|ac|)((1+\epsilon)^2-1)$, we get:

$$\begin{aligned} \left|a\bar{t}_0^2 + b\bar{t}_0 + c\right| &= \left(\bar{t}_0 \frac{2c}{-b+\sqrt{\hat{\Delta}}}\frac{b^2}{4c} + \bar{t}_0^2\frac{4ac}{4c}\right)((1+\epsilon)^2-1) \\ &\le (\bar{t}_0\tfrac{|b|}{2} + \bar{t}_0^2|a|)((1+\epsilon)^2-1). \end{aligned} \tag{10}$$

Now we can consider the rest of the process that calculates $\hat{t}_0$. Since both $-b$ and $\sqrt{\hat{\Delta}}$ are non-negative, the error associated with $-b+\sqrt{\hat{\Delta}}$ is bounded by $(-b+\sqrt{\hat{\Delta}})((1+\epsilon)^2-1)$. Let this error be e. From the previous analysis, we know that $|\bar{t}_0 - t_0| \le 3\epsilon^{\frac{1}{2}}$ and $|\hat{t}_0 - t_0| \le 3\epsilon^{\frac{1}{2}}$, so we assume $|\bar{t}_0| \le 1.1$ and $|\hat{t}_0| \le 1.1$, and we have:

$$\begin{aligned} \left|\frac{2c}{-b+\sqrt{\hat{\Delta}}+e} - \bar{t}_0\right| &\le \bar{t}_0 \left|1 - \frac{1}{1+\frac{e}{-b+\sqrt{\hat{\Delta}}}}\right| \\ &\le \bar{t}_0\left(\left|\frac{e}{-b+\sqrt{\hat{\Delta}}}\right| + O(\epsilon^2)\right) \le 2.2\epsilon + O(\epsilon^2). \end{aligned} \tag{11}$$

Therefore, $|\hat{t}_0 - \bar{t}_0| \le \left(\bar{t}_0 + 2.2\epsilon + O(\epsilon^2)\right)\epsilon + 2.2\epsilon + O(\epsilon^2) \le 3.4\epsilon$. As a result, $|a\hat{t}_0^2 + b\hat{t}_0 + c - a\bar{t}_0^2 - b\bar{t}_0 - c| \le |\hat{t}_0 - \bar{t}_0||\hat{t}_0 + \bar{t}_0||a| + |\hat{t}_0 - \bar{t}_0||b| \le (7.5|a| + 3.4|b|)\epsilon$. Together with Equation 10, we know that the residual error $|a\hat{t}_0^2 + b\hat{t}_0 + c|$ must be bounded by $(10|a| + 5|b|)\epsilon$.

Root t_1 when $b \le 0$. If $a \ne 0$, then t_1 exists. Let $\bar{t}_1 = \frac{-b+\sqrt{\hat{\Delta}}}{2a}$ be the root when the discriminant error is the only error. We have:

$$\begin{aligned} \left|a\bar{t}_1^2 + b\bar{t}_1 + c\right| &= \left|a\bar{t}_1^2 + b\bar{t}_1 + c - at_1^2 - bt_1 - c\right| \\ \le \left|\frac{(-b+\sqrt{\hat{\Delta}})^2}{4a} + \frac{b(-b+\sqrt{\hat{\Delta}})}{2a} - \frac{(-b+\sqrt{\Delta})^2}{4a} - \frac{b(-b+\sqrt{\Delta})}{2a}\right| &= \left|\frac{\hat{\Delta}-\Delta}{4a}\right|. \end{aligned} \tag{12}$$

To ensure that $t_1 \le 1$, we must have $|b| \le 2a$. Given $\left|\hat{\Delta} - \Delta\right| \le (b^2+4|ac|)((1+\epsilon)^2-1)$, we get:

$$\left|a\bar{t}_1^2 + b\bar{t}_1 + c\right| \le (\tfrac{|b|}{2} + |c|)((1+\epsilon)^2-1). \tag{13}$$

Since both $-b$ and $\sqrt{\hat{\Delta}}$ are still non-negative, the error associated with $-b+\sqrt{\hat{\Delta}}$ must be bounded by $(-b+\sqrt{\hat{\Delta}})((1+\epsilon)^2-1)$. Let this error be e. As before, we can safely assume $|\bar{t}_1| \leq 1.1$ and $|\hat{t}_1| \leq 1.1$, and we have:

$$\left|\frac{-b+\sqrt{\hat{\Delta}+e}}{2a} - \bar{t}_1\right| \leq \bar{t}_1 \left|\frac{e}{-b+\sqrt{\hat{\Delta}}}\right| \leq 2.2\epsilon + O(\epsilon^2). \tag{14}$$

Therefore, $|\hat{t}_1 - \bar{t}_1| \leq \left(\bar{t}_1 + 2.2\epsilon + O(\epsilon^2)\right)\epsilon + 2.2\epsilon + O(\epsilon^2) \leq 3.4\epsilon$. This means $|a\hat{t}_1^2 + b\hat{t}_1 + c - a\bar{t}_1^2 - b\bar{t}_1 - c| \leq |\hat{t}_1 - \bar{t}_1||\hat{t}_1 + \bar{t}_1||a| + |\hat{t}_1 - \bar{t}_1||b| \leq (7.5|a| + 3.4|b|)\epsilon$. Together with Equation 13, we know that the residual error $|a\hat{t}_1^2 + b\hat{t}_1 + c|$ is bounded by $(8|a| + 5|b| + 3|c|)\epsilon$.

Root t_1 when $b > 0$. This case is essentially the same as the first case when calculating t_0 under $b \leq 0$, except that the signs of b and c are flipped. So we should still have $|a\hat{t}_0^2 + b\hat{t}_0 + c| \leq (10|a| + 5|b|)\epsilon$.

In conclusion, if we set $\mathcal{B}$ as a upper bound on on $|a|$, $|b|$, and $|c|$, then we have $|a\hat{t}^2 + b\hat{t} + c| \leq 16\mathcal{B}\epsilon$, in which $\hat{t}$ is the computed root. This conclusion is still valid after we implement `Clamp` on $\hat{t}$. Because if not, then the quadratic function must be minimized somewhere between t and $\hat{t}$, given the fact that $a \geq 0$. The distance between this minimum time $t_{\min}$ and t is: $\frac{\sqrt{\Delta}}{2a}$. Since $|\hat{t} - t| \leq 3\epsilon^{\frac{1}{2}}$, we have: $\sqrt{\Delta} \leq 6a\epsilon^{\frac{1}{2}}$. However, the magnitude of the function minimum is $\frac{\Delta}{4a} \leq 9a\epsilon$, which means $|at_{\min}^2 + bt_{\min} + c| \leq 16\mathcal{B}\epsilon$ even at the minimum. Since the quadratic function is monotonic in $[t, t_{\min}]$ and $[t_{\min}, \hat{t}]$, we must have $|as^2 + bs + c| \leq 16\mathcal{B}\epsilon$ for any $s \in [t, \hat{t}]$. This contradicts our previous assumption. Therefore, $|a\hat{t}^2 + b\hat{t} + c| \leq 16\mathcal{B}\epsilon$ is valid even after clamping $\hat{t}$ to 0 or 1.

A.3 Degenerate Cases

Our quadratic solver can robustly handle degenerate cases and the error bounds provided in Appendix A.1 and A.2 are still valid.

- If $a = b = 0$, we must have $c = 0$ and the quadratic equation becomes trivial. In that case, the solver simply returns $t = 0$ as the only root.
- If $a = 0$ and $b \neq 0$, only one root exists and we compute it as $t_0 = \frac{c}{-b}$. Since it has only one step and $t_0 \in [0, 1]$, we have $|\hat{t}_0 - t_0| \leq \epsilon$ and $|a\hat{t}_0^2 + b\hat{t}_0 + c| \leq \mathcal{B}\epsilon$. So the error bounds in Appendix A.1 and A.2 are still valid.
- If $a \neq 0$ and $b = 0$ and $c = 0$, then $t_0 = t_1 = 0$ and there is no root error or residual error. If $a \neq 0$ and $b = 0$ and $c \neq 0$, then $\Delta \neq 0$. If $-b + \sqrt{\Delta}$ exists, it must be nonzero and its computed version must be nonzero as well. So all of the formulae in Equation 4 can be computed and the analysis is still valid.
- Finally, when $a \neq 0$ and $b \neq 0$, we must have $-b + \sqrt{\Delta} > 0$ when $b < 0$, and $-b - \sqrt{\Delta} < 0$ when $b > 0$. Since b is a floating-point number, the computed version of $-b \pm \sqrt{\Delta}$ is nonzero as well and all of the formulae in Equation 4 can be computed. If $a \neq 0$, $b \neq 0$, and $c = 0$, we have $t_0 = 0$ when $b \leq 0$ and $t_1 = 0$ when $b > 0$. In that case, there is no residual error and the analysis in Appendix A.2 is still valid.

B The Cubic Solver

Given a cubic equation $F(t) = at^3 + bt^2 + ct + d = 0$, the cubic solver first splits the interval $[0, 1]$ into a set of sub-intervals by computing the local minimum and maximum using the quadratic solver, and then uses the hybrid Newton-bisection method to find the root in each sub-interval. If the cubic equation is degenerate ($a = b = c = d = 0$), the root can be trivially reported as $t = 1$, and the vertex-triangle and edge-edge CCD algorithms will resort to vertex-edge CCD to find the actual collision time. Otherwise, there are a finite number of roots and there can be at most one root in every sub-interval. In this appendix, we will study the errors in the formulation of the sub-intervals first, and then we will discuss how to make the solver robust against the errors.

Local Minimum and Maximum. The local minimum and maximum are computed by solving the quadratic equation $et^2 + ft + g = 0$, in which $e = 3a$, $f = 2b$, and $g = c$. According to Subsection 4.4 and 4.5, the computation of the coefficients involves errors and we have: $|\hat{e} - e| \leq 18B^3((1+\epsilon)^6 - 1) \leq 109B^3\epsilon$, $|\hat{f} - f| \leq 36B^3((1+\epsilon)^7 - 1) \leq 253B^3\epsilon$, and $|\hat{g} - g| \leq 18B^3((1+\epsilon)^7 - 1) \leq 127B^3\epsilon$. If the local minimum or maximum does not exist, then the computation here will cause unnecessary sub-intervals and computational costs, which fortunately has no influence on the outcome of the whole algorithm.

Let t_{m} be the exact root of $et^2 + ft + g = 0$, $\bar{t}_{\mathrm{m}}$ be the exact root of $\hat{e}t^2 + \hat{f}t + \hat{g} = 0$, and $\hat{t}_{\mathrm{m}}$ be the computed version of $\bar{t}_{\mathrm{m}}$. Without loss of generality, we assume $e \geq 0$. There are three possible cases.

Case 1. When $|e| \geq |f|$ and $|e| > 8000B^3\epsilon$, we must have $|g| \leq |e| + |f| \leq 2|e|$ to ensure the existence of t_{m} in $[0, 1]$. Let $\Delta = f^2 - 4eg$ and $\bar{\Delta} = \mathtt{Clamp}(\hat{f}^2 - 4\hat{e}\hat{g}, 0, +\infty)$ be two discriminants. Using the error bounds provided previously, we have:

$$\begin{gathered}\left|\bar{\Delta} - \Delta\right| \leq \left|\hat{f} + f\right|\left|\hat{f} - f\right| + 4\,|\hat{e}|\,|\hat{g} - g| + 4\,|g|\,|\hat{e} - e| \\ \leq (2|e| + 253B^3\epsilon) \cdot 253B^3\epsilon + 4(|e| + 109B^3\epsilon) \cdot 127B^3\epsilon + \\ 8|e| \cdot 109B^3\epsilon \leq 1886|e|B^3\epsilon + 119381B^6\epsilon^2.\end{gathered} \tag{15}$$

Since $t_{\mathrm{m}} \in [0, 1]$, $\hat{e} \neq 0$, and $|\sqrt{\bar{\Delta}} - \sqrt{\Delta}| \leq \sqrt{|\bar{\Delta} - \Delta|}$, We can get an upper bound on $|\bar{t}_{\mathrm{m}} - t_{\mathrm{m}}|$ as:

$$\frac{|\hat{f} - f| + |\sqrt{\bar{\Delta}} - \sqrt{\Delta}|}{|2\hat{e}|} + t_{\mathrm{m}}\frac{|\hat{e} - e|}{|\hat{e}|} \leq \frac{\sqrt{1886|e|B^3\epsilon} + 817B^3\epsilon}{2|\hat{e}|}. \tag{16}$$

The solver defines the initial interval as $[0, 1]$, which is equivalent to defining 0 and 1 as two default roots. So if $\bar{t}_{\mathrm{m}} > 1$ or $\bar{t}_{\mathrm{m}} < 0$, we assume $\hat{t}_{\mathrm{m}} = \bar{t}_{\mathrm{m}} = 0$ or 1, and the upper bound in Equation 16 still holds for $|\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}|$. Otherwise, $\bar{t}_{\mathrm{m}} \in (0, 1)$, we know from Appendix A.1 that $\left|\hat{t}_{\mathrm{m}} - \bar{t}_{\mathrm{m}}\right| \leq 3\epsilon^{\frac{1}{2}}$ due to the quadratic solver. According to Taylor expansion, $\left|F(\hat{t}_{\mathrm{m}}) - F(t_{\mathrm{m}})\right|$ must be bounded by:

$$|3at_{\mathrm{m}}^2 + 2bt_{\mathrm{m}} + c|\left|\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}\right| + |3at_{\mathrm{m}} + b|\left|\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}\right|^2 + |a|\left|\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}\right|^3. \tag{17}$$

Since $3at_{\mathrm{m}}^2 + 2bt_{\mathrm{m}} + c = 0$, $|3at_{\mathrm{m}} + b| \leq 1.5|e|$, and $|a||\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}| \leq \frac{1}{3}|e|$, We can further bound $\left|F(\hat{t}_{\mathrm{m}}) - F(t_{\mathrm{m}})\right|$ by:

$$\begin{gathered}2|e|\left(\frac{1886|e|B^3\epsilon + (817B^3\epsilon)^2}{4\hat{e}^2} + 9\epsilon\right) \cdot 3 \leq 2955B^3\epsilon\left(\frac{e}{\hat{e}}\right)^2 + 972B^3\epsilon \\ \leq 2955B^3\epsilon\left(1 + \frac{109}{8000 - 109}\right)^2 + 972B^3\epsilon \leq 4010B^3\epsilon.\end{gathered} \tag{18}$$

Case 2. When $|f| \geq |e|$ and $|f| > 8000B^3\epsilon$, we must have $|g| \leq 2|f|$ to ensure the existence of t_{m} in $[0, 1]$. Similar to Case 1, we have $|\bar{\Delta} - \Delta| \leq 1886|f|B^3\epsilon + 119381B^6\epsilon^2$ here. If $f > 0$, the only valid solution in $[0, 1]$ is: $t_{\mathrm{m}} = (2g)/(-f - \sqrt{\Delta})$. Since $|f| > 8000B^3\epsilon$, we get $\hat{f} > 0$ and $-\hat{f} - \sqrt{\bar{\Delta}} \neq 0$. We can then bound $|\bar{t}_{\mathrm{m}} - t_{\mathrm{m}}|$ by:

$$\frac{2|\hat{g} - g|}{\hat{f} + \sqrt{\bar{\Delta}}} + t_{\mathrm{m}}\frac{|\hat{f} - f| + |\sqrt{\bar{\Delta}} - \Delta|}{\hat{f} + \sqrt{\bar{\Delta}}} \leq \frac{\sqrt{1886|f|B^3\epsilon} + 853B^3\epsilon}{\hat{f}}. \tag{19}$$

When $\bar{t}_{\mathrm{m}} > 1$ or $\bar{t}_{\mathrm{m}} < 0$, we assume $\hat{t}_{\mathrm{m}} = \bar{t}_{\mathrm{m}} = 0$ or 1, and the upper bound in Equation 19 still holds for $|\hat{t}_{\mathrm{m}} - t_{\mathrm{m}}|$. Otherwise, $\bar{t}_{\mathrm{m}} \in (0, 1)$, we know from Appendix A.1 that $\left|\hat{t}_{\mathrm{m}} - \bar{t}_{\mathrm{m}}\right| \leq 3\epsilon^{\frac{1}{2}}$

due to the quadratic solver. Using Taylor expansion, we can bound $\left|F(\hat{t}_m) - F(t_m)\right|$ by:

$$2|f|\left(\frac{1886|f|B^3\epsilon+(853B^3\epsilon)^2}{f^2}+9\epsilon\right)\cdot 3 \le 11862B^3\epsilon\left(\frac{f}{\hat{f}}\right)^2 + 1944B^3\epsilon$$
$$\le 11862B^3\epsilon\left(1+\frac{253}{8000-253}\right)^2 + 1944B^3\epsilon \le 14600B^3\epsilon. \tag{20}$$

On the other hand, if $f < 0$, the only valid solution is: $t_m = (2g)/(-f+\sqrt{\Delta})$. The same analysis applies and we also have: $\left|F(\hat{t}_m) - F(t_m)\right| \le 14600B^3\epsilon$.

Case 3. Finally, $|e| \le 8000B^3\epsilon$ and $|f| \le 8000B^3\epsilon$. In this case, the computation of $\hat{t}_m$ does not matter. Based on the fact that $|\hat{t}_m - t_m| \le 1$, we apply Taylor expansion to bound $\left|F(\hat{t}_m) - F(t_m)\right|$ by:

$$|3at_m + b|\left|\hat{t}_m - t_m\right|^2 + |a|\left|\hat{t}_m - t_m\right|^3 \le 16000B^3\epsilon. \tag{21}$$

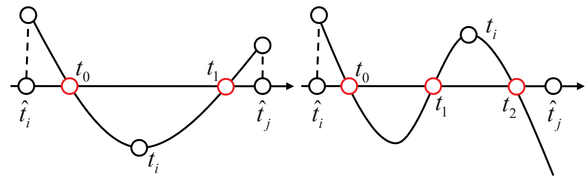
From the three cases cases, we see $\left|F(\hat{t}_m) - F(t_m)\right| \le 16000B^3\epsilon$. It is also straightforward to see that $|F(t) - F(t_m)| \le 16000B^3\epsilon$, for any $t \in [t_m, \hat{t}_m]$ as well.

The convergence criterion. For every root t_0, the goal of the cubic solver is to find a computed root $\hat{t}_0$, such that $F(t) \le \mu$ for any $t \in [t_0, \hat{t}_0]$, in which μ is a user-specified threshold. If the sub-intervals can be computed exactly, then the cubic function changes monotonically and at most one root can exist in each sub-interval. Once the solver finds a solution $\hat{t}_0$ such that $F(\hat{t}_0) \le \mu$, then $F(t) \le \mu$ is satisfied for any t in-between as well.

The problem is that the computation of $F(t)$ may contain an error. Let $\bar{F}(t) = ((\hat{a}t + \hat{b})t + \hat{c})t + \hat{d}$, in which $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ are the computed variables, and $\hat{F}(t)$ be the computed version of $\bar{F}(t)$. There are six arithmetic operations in $F(t)$. From Subsection 4.4 and Theorem 4.1, we know $\left|\hat{F}(t) - F(t)\right| \le \left|\bar{F}(t) - F(t)\right| + \left|\hat{F}(t) - \bar{F}(t)\right| \le 313B^3\epsilon + 4\mathcal{B}((1+\epsilon)^6 - 1) \le 746B^3\epsilon$, in which $\mathcal{B} = 18B^3(1+\epsilon)^7$ is an upper bound on $|\hat{a}|$, $|\hat{b}|$, $|\hat{c}|$, and $|\hat{d}|$. So to ensure that $|F(\hat{t}_0)| \le \mu$, we need to adjust the convergence threshold from μ to $\hat{\mu}$, for $\hat{\mu} \le \mu - 746B^3\epsilon$. Once we find $|\hat{F}(\hat{t}_0)| \le \hat{\mu}$, we know $|F(\hat{t}_0)| \le \mu$ must also be true.

The next issue is how to evaluate the existence of the root. The cubic solver uses the signs of $F(t_i)$ and $F(t_j)$ to determine whether a roots exists in a sub-interval $[t_i, t_j]$. If $F(t_i)$ and $F(t_j)$ have different signs while $\hat{F}(t_i)$ and $\hat{F}(t_j)$ have the same sign, the solver will fail even before the iterative process starts. This means $\hat{\mu}$ must also be greater than $746B^3\epsilon$. If $|\hat{F}(t_i)| \le \hat{\mu}$ or $|\hat{F}(t_j)| \le \hat{\mu}$, the solver reports a solution at t_i or t_j immediately. Otherwise, if $\hat{F}(t_i) > \hat{\mu}$, we know $F(t_i) > 0$; and if $\hat{F}(t_i) < -\hat{\mu}$, we know $F(t_i) < 0$. We can avoid or continue the process according to the signs. The continuity of the function ensures that for every computed root $\hat{t}_0$, there exists an exact root t_0, such that the function changes monotonically from t_0 to $\hat{t}_0$. So if sub-intervals are exact, the solver can detect every sub-interval that contains a root and find the root in it.

Unfortunately, the sub-intervals may not be exact and there can be more than one exact roots within a computed sub-interval. If such a event happens, we must ensure that the computed root is valid for all the exact roots. If there are two exact roots in a computed sub-interval $[\hat{t}_i, \hat{t}_j]$ as Figure 8a shows, then there must exist a local minimum or maximum between the two roots t_0 and t_1. Without loss of generality, if this local minimum/maximum is t_i, whose computed time is $\hat{t}_i$, we know $|F(\hat{t}_i) - F(t_i)| \le 16000B^3\epsilon$ from the previous analysis on local minimum/maximum. Since only one root exists in $[\hat{t}_i, t_i]$, $F(\hat{t}_i)$ and $F(t_i)$ must have different signs and we have $|F(t)| \le 16000B^3\epsilon$ for any t in $[\hat{t}_i, t_1]$. So if we set the actual convergence threshold to $\hat{\mu} \ge (16000 + 746)B^3\epsilon = 16746B^3\epsilon$,



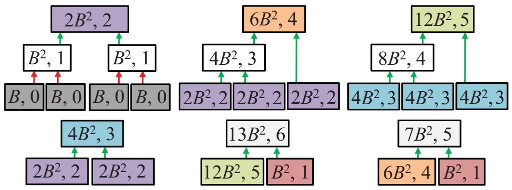
(a) Two root case (b) Three root case

Figure 8: *Computed sub-intervals. More than one roots may exist in a computed sub-interval. To address this problem, the convergence threshold μ needs to be sufficiently enlarged.*

we know $\hat{t}_i$ can be detected as a solution for both t_0 and t_1. If the local minimum/maximum is t_k, whose solution $\hat{t}_k$ is outside of $[\hat{t}_i, \hat{t}_j]$, then either $\hat{t}_i$ or $\hat{t}_j$ must be within $[t_k, \hat{t}_k]$. If so, we can simply treat $\hat{t}_i$ or $\hat{t}_j$ as $\hat{t}_k$ and we have $|F(\hat{t}_i) - F(t_k)| \le 16000B^3\epsilon$ (or $|F(\hat{t}_j) - F(t_k)| \le 16000B^3\epsilon$). Therefore, $\hat{\mu} \ge 16746B^3\epsilon$ is still a sufficient condition to ensure the detection of both roots. Similar analysis can be performed on the cases when three roots exist in a computed sub-interval, except that it is possible to have two roots from t_i to $\hat{t}_i$, as Figure 8b shows. Since $|F(t) - F(t_i)| \le 16000B^3\epsilon$ for any $t \in [\hat{t}_i, t_i]$, we know $F(\hat{t}_i) \le 32000B^3\epsilon$ and we use $\hat{\mu} \ge (32000 + 746)B^3\epsilon = 32746B^3\epsilon$ to detect $\hat{t}_i$ as a valid root.

In summary, the solver first tests $|\hat{F}(\hat{t}_i)| \ge \hat{\mu}$ and $|\hat{F}(\hat{t}_j)| \ge \hat{\mu}$. If both are satisfied and $\hat{F}(\hat{t}_i)\hat{F}(\hat{t}_j) < 0$, it means there can be at most one root within $[\hat{t}_i, \hat{t}_j]$ and the solver will start the iterative process. Otherwise, the solver will report $\hat{t}_i$ as a root if $|\hat{F}(\hat{t}_i)| \le \hat{\mu}$, or $\hat{t}_j$ if $|\hat{F}(\hat{t}_j)| \le \hat{\mu}$. The solver may report them both, if both conditions are satisfied. Since $32746B^3\epsilon \le \hat{\mu} \le \mu - 746B^3\epsilon$, we must set $\mu \ge 33492B^3\epsilon$. This condition is satisfied automatically when using $\mu = 64B^3\epsilon^{\frac{3}{4}}$ for both vertex-triangle and edge-edge collision cases. In practice, we simplify the computation by using $\hat{\mu} = 64B^3\epsilon^{\frac{3}{4}}$ directly, assuming that $\mu = 128B^3\epsilon^{\frac{3}{4}}$. The analysis and the conclusions provided in Subsection 4.4 and 4.5 are still valid.

The convergence. To guarantee the robustness of our algorithm, we need to know whether the cubic solver always converges when using floating-point numbers. To understand this problem, we first present an important property of floating-point arithmetic. Let a and b be two unique floating-point numbers. We claim that if $\texttt{Round}((a+b)/2) = a$ or $\texttt{Round}((a+b)/2) = b$, then there can be at most one floating-point number between a and b. Assuming that $a < b$, we first consider the case when $\texttt{Round}((a+b)/2) = a$. Let a_1 and a_2 be the two floating-point numbers immediately above a: $a < a_1 < a_2$. If $\texttt{Round}((a+b)/2) = a$, we must have $\frac{a+b}{2} < a_1$ and $b < 2a_1 - a$. Since $a_1 - a \le a_2 - a_1$, we have $b < a_2$, so $b = a$ or $b = a_1$, and no floating-point number exists between a and b. Similarly, let b_1, b_2, b_3 be the three floating-point numbers immediately below b: $b_3 < b_2 < b_1 < b$. If $\texttt{Round}((a+b)/2) = b$, we have $\frac{a+b}{2} > b_1$ and $2b_1 - b < a$. Since $b_1 - b_3 \ge b - b_1$, we have $a > b_3$. So we must have $a = b$, $a = b_1$, or $a = b_2$. Therefore, at most one floating-point number can exist between a and b. This property indicates that the Newton-Bisection method can keep subdividing the interval until the interval spans no more than three floating-point numbers. When floating-point numbers are in $[0, 1]$, the gap between two adjacent ones cannot exceed ϵ. Therefore, the ultimately subdivided interval size must be bounded by 2ϵ. Let $[t_0, t_1]$ be an interval that contains an exact root t. Using Taylor expansion, we know that both $|F(t_0)|$ and $|F(t_1)|$ should be bounded by: $\left|3at^2 + 2bt + c\right|2\epsilon + |3at + b|(2\epsilon)^2 + |a|(2\epsilon)^3 \le 13\mathcal{B}\epsilon$, in which $\mathcal{B} = 18B^3(1+\epsilon)^7$. Together with the sign detection error $746B^3\epsilon$, we must have $\hat{\mu} \ge 981B^3\epsilon$ to ensure the convergence of the cubic solver. Fortunately, this has already been satisfied previously.



(a) $\mathbf{x}_a$, $\mathbf{x}_b$, and $\mathbf{x}_c$ (b) Value a and b (c) Value c

Figure 9: *Error bounds in different cases. The red arrows represent* `multiply` *and the green arrows represent* `add` *or* `subtract`*.*

C Errors in Vertex-Edge CCD

In this section, we provide detailed error analysis on the first four cases in vertex-edge CCD. Our idea is to formulate a new function $G_{0ij}(t)$ from $F_{0ij}(t)$, by removing any component in $\mathbf{x}_a$ or $\mathbf{x}_b$ if it is less than $\rho = 3B^2\epsilon^{\frac{1}{2}}$. Let t_0^{F} be the desired minimum of $F_{0ij}(t)$, t_0 be the minimum of $G_{0ij}(t)$ closest to t_0^{F}, and $\hat{t}_0$ be its computed version. Our ultimate goal is to find an upper bound on $|F_{0ij}(\hat{t}_0) - F_{0ij}(t_0^{\mathrm{F}})|$. Firstly, we formulate an upper bound on $|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)|$.

Case 1. In Case 1, $t_0 = \hat{t}_0 = 0$ or 1. So $|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)| = 0$.

Case 2. If the three components of $\mathbf{x}_{ji} + t\mathbf{v}_{ji}$ have unique magnitudes and no component in $(\mathbf{x}_{0i}+t\mathbf{v}_{0i})\times(\mathbf{x}_{ji}+t\mathbf{v}_{ji})$ is zero, then $G_{0ij}(t)$ can be considered locally at t_0 as a quadratic function $G_{0ij}(t) = at^2+bt+c$ as Subsection 3.2 shows. Let $\bar{G}_{0ij}(t) = \hat{a}t^2+\hat{b}t+\hat{c}$. Using Theorem 4.1 and Figure 9a, we get $\|\hat{\mathbf{x}}_a - \mathbf{x}_a\|_\infty \le 2B^2((1+\epsilon)^2-1)$, $\|\hat{\mathbf{x}}_b - \mathbf{x}_b\|_\infty \le 4B^2((1+\epsilon)^3-1)$, $\|\hat{\mathbf{x}}_c - \mathbf{x}_c\|_\infty \le 2B^2((1+\epsilon)^2-1)$. Suppose that $S \le B$, we can further get $|\hat{a} - a| \le 6B^2((1+\epsilon)^4-1)$, $|\hat{b} - b| \le 13B^2((1+\epsilon)^6-1)$, and $|\hat{c} - c| \le 7B^2((1+\epsilon)^5-1)$, as Figure 9b and 9c shows. To make $t_0 = \frac{-b}{2a} \in [0,1]$, we must have $a \ne 0$ and $|b| \le |2a| \le |2\hat{a}| + 48B^2\epsilon + B^2O(\epsilon^2)$. So when $|\hat{a}| \le 25B^2\epsilon$, we have $|a| \le 49B^2\epsilon + B^2O(\epsilon)$, $|b| \le 98B^2\epsilon + B^2O(\epsilon)$, and $|\hat{b}| \le 176B^2\epsilon + B^2O(\epsilon)$. Given the fact that $t_0 \in [0,1]$ and $\hat{t}_0 \in [0,1]$, we have:

$$|\bar{G}_{0ij}(\hat{t}_0)-G_{0ij}(t_0)| \le |\hat{a}|+|a|+|\hat{b}-b||\hat{t}_0|+|b||\hat{t}_0-t_0|+|\hat{c}-c| \le 286B^2\epsilon. \tag{22}$$

Alternatively, if $|\hat{a}| > 25B^2\epsilon$, we have:

$$\begin{aligned} \left|\tfrac{-\hat{b}^2}{4\hat{a}} - \tfrac{-b^2}{4a}\right| &\le \left|\hat{b} - b\right| \tfrac{\left|\hat{b}-b\right|+2|b|}{|4\hat{a}|} + \left|\tfrac{b}{4a}\right|\left|\tfrac{b(a-\hat{a})}{\hat{a}}\right| \\ &\le \tfrac{(78B^2\epsilon+B^2O(\epsilon^2))(174B^2\epsilon+B^2O(\epsilon^2)+|4\hat{a}|)}{|4\hat{a}|} + \tfrac{(|2\hat{a}|+48B^2\epsilon+B^2O(\epsilon^2))(24B^2\epsilon+B^2O(\epsilon^2))}{2|\hat{a}|} \\ &\le 264B^2\epsilon. \end{aligned} \tag{23}$$

Let $\tilde{t}_0 = -\hat{b}/(2\hat{a})$, we get:

$$|\bar{G}_{0ij}(\tilde{t}_0) - G_{0ij}(t_0)| \le \left|\tfrac{-\hat{b}^2}{4\hat{a}} - \tfrac{-b^2}{4a}\right| + |\hat{c} - c| \le 300B^2\epsilon. \tag{24}$$

In practice, $\tilde{t}_0$ is set to 0 if it is less than 0. The existence of t_0 in $[0,1]$ minimizing the quadratic form requires $a > 0$ and $b \le 0$. According to $|\hat{a}| > 25B^2\epsilon$, we have $\hat{a} > 0$ and $\hat{b} > 0$. Since $|\hat{b} - b| \le 78B^2\epsilon + B^2O(\epsilon^2)$, we must have $-79B^2\epsilon \le b \le 0$. So,

$$|\bar{G}_{0ij}(0) - G_{0ij}(t_0)| \le \left|\tfrac{b^2}{4a}\right| + |\hat{c} - c| \le \left|\tfrac{b}{2}\right| + |\hat{c} - c| \le 75B^2\epsilon. \tag{25}$$

Alternatively, $\tilde{t}_0$ is set to 1 if it is greater than 1. In that case, we have $-b \le 2a$ and $2\hat{a} \le -\hat{b}$. Since $|\hat{a} - a| \le 24B^2\epsilon + B^2O(\epsilon^2)$ and $|\hat{b} - b| \le 78B^2\epsilon + B^2O(\epsilon^2)$, we get $2a - 127B^2\epsilon \le -b \le 2a$. So,

$$\begin{aligned} |\bar{G}_{0ij}(1) - G_{0ij}(t_0)| &\le \left|\hat{a} + \hat{b} + \tfrac{b^2}{4a}\right| + |\hat{c} - c| \\ &\le |\hat{a} - a| + |\hat{b} - b| + (2a + b)\left(\left|\tfrac{2a}{4a}\right| + \left|\tfrac{b}{4a}\right|\right) + |\hat{c} - c| \le 265B^2\epsilon. \end{aligned} \tag{26}$$

Equation 25 and 26 indicate that $|\bar{G}_{0ij}(\tilde{t}_0) - G_{0ij}(t_0)| \le 300B^2\epsilon$ is still valid after clamping $\tilde{t}_0$. Now let us examine the `divide` step that causes errors between $\tilde{t}_0$ and $\hat{t}_0$, if $\tilde{t}_0$ is not clamped. Since $0 \le \tilde{t}_0 \le 1$, we have $|\hat{t}_0 - \tilde{t}_0| \le |\tilde{t}_0|\epsilon \le \epsilon$. So we get $|\bar{G}_{0ij}(\hat{t}_0) - \bar{G}_{0ij}(\tilde{t}_0)| \le (2+\epsilon)|\hat{a}|\epsilon+|\hat{b}|\epsilon \le 26B^2\epsilon$ and $|\bar{G}_{0ij}(\hat{t}_0)-G_{0ij}(t_0)| \le 326B^2\epsilon$. Meanwhile, by the definition of $\bar{G}_{0ij}$, we have $|\bar{G}_{0ij}(\hat{t}_0) - G_{0ij}(\hat{t}_0)| \le (24 + 78 + 35 + 1)B^2\epsilon = 138B^2\epsilon$. Together, $|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)| \le (326 + 138)B^2\epsilon \le 464B^2\epsilon$.

We note that $|G_{0ij}(t) - G_{0ij}(t_0)| \le 464B^2\epsilon$ is valid for any $t \in [t_0, \hat{t}_0]$. This is because the transition from G_{0ij} to $\bar{G}_{0ij}$ can be treated as a continuous functional. When $t \in [t_0, \tilde{t}_0]$, there must exist an intermediate function $G_{0ij}^{\mathrm{int}}(t) = a^{\mathrm{int}}t^2 + b^{\mathrm{int}}t + c^{\mathrm{int}}$ whose exact solution (after clamping) is t. By treating G_{0ij}^{int} as $\bar{G}_{0ij}$ and t as $\tilde{t}$, we can use the same analysis to reach this conclusion. If $t \in [\tilde{t}_0, \hat{t}_0]$ instead, we treat t as $\hat{t}_0$ and $|G_{0ij}(t) - G_{0ij}(t_0)| \le 464B^2\epsilon$ must be true as well.

Case 3. In Case 3, t_0 is found by solving a linear equation. Without loss of generality, we suppose that $x_{ji} + tu_{ji} = y_{ji} + tv_{ji}$. The error $|\hat{t}_0 - t_0|$ here can be caused by `subtract` and `divide` only. Let $a = u_{ji} - v_{ji}$ and $b = y_{ji} - x_{ji}$, such that $t_0 = b/a$ and $\hat{t}_0$ be its computed version. The existence of t_0 requires $a \ne 0$. Assuming that the velocities are substantially larger than the smallest floating-point value, so there is no underflow and $\hat{a} \ne 0$ as well. Given $|\hat{a} - a| \le |a|\epsilon$, $|\hat{b} - b| \le |b|\epsilon$ and $|b| \le |a|$, we get:

$$\left|\tfrac{\hat{b}}{\hat{a}} - \tfrac{b}{a}\right| \le \tfrac{|b(\hat{a}-a)-a(\hat{b}-b)|}{|a\hat{a}|} \le 3\epsilon. \tag{27}$$

Using $t_0 \in [0,1]$, we have $\left|\hat{t}_0 - t_0\right| \le (1+3\epsilon)\epsilon+3\epsilon \le 5\epsilon$. If $\hat{t}_0 < 0$ or $\hat{t}_0 > 1$, the algorithm clamps it to 0 or 1. Doing this makes $\hat{t}_0$ closer to t_0, so $|\hat{t}_0 - t_0| \le 5\epsilon$ must still be valid. Using the bounds derived in Case 2, we can bound $\left|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)\right|$ by:

$$3\left(2B^2\left|\hat{t}_0^2 - t_0^2\right| + 4B^2\left|\hat{t}_0 - t_0\right|\right) + B^2\left|\hat{t}_0 - t_0\right| \le 126B^2\epsilon. \tag{28}$$

This conclusion is valid for any t in $[t_0, \hat{t}_0]$, which can be derived from Equation 28 after replacing $\hat{t}_0$ by t.

Case 4. Case 4 is the most complex one among four. Without loss of generality, we suppose that $Q(t_0) = x_at_0^2 + x_bt_0 + x_c$ ($x_a \ge 0$) is the quadratic function whose root gives the collision time t_0. We first study $\bar{Q}(\bar{t}_0) = \hat{x}_a\bar{t}_0^2 + \hat{x}_b\bar{t}_0 + \hat{x}_c$, whose exact root provided by the quadratic solver in Appendix A.1 is $\bar{t}_0$ (without clamping). Since $G_{0ij}(t)$ is formulated in a way that $|x_a|$ and $|x_b|$ are greater than $3B^2\epsilon^{\frac{1}{2}}$, there are two possible cases.

- **Case 4.1.** When $5x_a \ge |x_b| \ge 3B^2\epsilon^{\frac{1}{2}}$, we must have $|x_c| \le |x_a| + |x_b| \le 6|x_a|$ to ensure the existence of t_0 in $[0,1]$. Let $\Delta = x_b^2 - 4x_ax_c$ be the discriminant of $Q(t_0)$ and $\bar{\Delta} = \mathtt{Clamp}(\hat{x}_b^2 - 4\hat{x}_a\hat{x}_c, 0, +\infty)$ be the discriminant of $\bar{Q}(\bar{t}_0)$. Using the error bounds provided in Case 2, we have $\left|\bar{\Delta} - \Delta\right| \le (2|x_b|\cdot 12 + 4|x_a|\cdot 4 + 4|x_c|\cdot 4)B^2\epsilon + B^4O(\epsilon^2) \le 232|x_a|B^2\epsilon + B^4O(\epsilon^2)$. So $|\sqrt{\bar{\Delta}} - \sqrt{\Delta}| \le \sqrt{232x_a}B\epsilon^{\frac{1}{2}} + B^2O(\epsilon)$. Since $x_a \ge 3B^2\epsilon^{\frac{1}{2}}$, $\hat{x}_a > 0$ and we can get an upper bound on $|\bar{t}_0 - t_0|$:

  $$\tfrac{|\hat{x}_b-x_b|+|\sqrt{\bar{\Delta}}-\sqrt{\Delta}|}{2|\hat{x}_a|} + |t_0|\tfrac{|\hat{x}_a-x_a|}{|\hat{x}_a|} \le \tfrac{\sqrt{232x_a}B\epsilon^{\frac{1}{2}}+B^2O(\epsilon)}{2(x_a-5B^2\epsilon)} \le 5\epsilon^{\frac{1}{4}}. \tag{29}$$

  If $\bar{\Delta}$ is not clamped, $\bar{t}_0$ is a valid root and $\bar{Q}(\bar{t}_0) = 0$. If $\bar{\Delta}$ is clamped, then $\Delta \ge 0$ while $\hat{x}_b^2 - 4\hat{x}_a\hat{x}_c < 0$. From the previous analysis, we know that $\hat{x}_b^2 - 4\hat{x}_a\hat{x}_c \ge -233|x_a|B^2\epsilon$. According to the quadratic formula in Appendix A.1, when $\bar{\Delta}$ is clamped, $\bar{Q}(\bar{t}_0) = \frac{-\hat{x}_b^2+4\hat{x}_a\hat{x}_c}{4x_a} \le 60B^2\epsilon$.

- **Case 4.2.** When $|x_b| \geq 5x_a \geq 3B^2\epsilon^{\frac{1}{2}}$, we must have $|x_c| \leq 1.2|x_b|$ to ensure that t_0 is in $[0, 1]$. We have $\left|\bar{\Delta} - \Delta\right| \leq (2|x_b| \cdot 12 + 4|x_a| \cdot 4 + 4|x_c| \cdot 4)B^2\epsilon + B^4O(\epsilon^2) \leq 47|x_b|B^2\epsilon + B^4O(\epsilon^2)$ and $|\sqrt{\bar{\Delta}} - \sqrt{\Delta}| \leq \sqrt{47|x_b|}B\epsilon^{\frac{1}{2}} + B^2O(\epsilon)$. If $x_b > 0$, the only valid solution is: $t = (2x_c)/(-x_b - \sqrt{\Delta})$. Since $|x_b| \geq 3B^2\epsilon^{\frac{1}{2}}$, $-\hat{x}_b - \sqrt{\bar{\Delta}} \neq 0$ and $|\bar{t}_0 - t_0|$ is bounded by:

$$\frac{2|\hat{x}_c - x_c|}{\hat{x}_b + \sqrt{\bar{\Delta}}} + |t_0| \frac{|\hat{x}_b - x_b| + |\sqrt{\bar{\Delta}} - \sqrt{\Delta}|}{\hat{x}_b + \sqrt{\bar{\Delta}}} \leq \frac{\sqrt{47|x_b|}B\epsilon^{\frac{1}{2}} + B^2O(\epsilon)}{|x_b| - 13B^2\epsilon} \leq 5\epsilon^{\frac{1}{4}}. \quad (30)$$

Similarly, if $x_b < 0$, the only valid solution is $t = (2x_c)/(-x_b + \sqrt{\Delta})$ and $|\bar{t}_0 - t_0| \leq 5\epsilon^{\frac{1}{4}}$ is true again.

Additionally, we have $\Delta = x_b^2 - 4x_ax_c \geq x_b^2 - 4 \cdot 0.2x_b \cdot 1.2x_b = 0.04x_b^2$. Since $|x_b| \geq 3B^2\epsilon^{\frac{1}{2}}$ while $\left|\bar{\Delta} - \Delta\right| \leq 48|x_b|B^2\epsilon$, we must have $\bar{\Delta} > 0$ and it is not clamped. Therefore, $\bar{t}_0$ is actually a valid root and $\bar{Q}(\bar{t}_0) = 0$.

The above analysis indicates that although $|\bar{t}_0 - t_0| \leq 5\epsilon^{\frac{1}{4}}$, we still have: $|Q(\bar{t}_0)| \leq |\bar{Q}(\bar{t}_0)| + |\bar{Q}(\bar{t}_0) - Q(\bar{t}_0)| \leq (60 + 4 + 12 + 4)B^2\epsilon + B^2O(\epsilon^{\frac{5}{4}}) \leq 81B^2\epsilon$. In addition, any $t \in [t_0, \bar{t}_0]$ can be considered as the exact quadratic solution of an intermediate function $Q^{\text{int}}(t)$ between $Q(t)$ and $\bar{Q}(t)$. By treating t as $\bar{t}_0$ and $Q^{\text{int}}(t)$ as $\bar{Q}(t)$, we know $|Q(t)| \leq 81B^2\epsilon$ for any $t \in [t_0, \bar{t}_0]$. According to Taylor expansion, we know $|Q(\bar{t}_0)| = \left|\frac{1}{2}Q''(t_0)(\bar{t}_0 - t_0)^2 + Q'(t_0)(\bar{t}_0 - t_0)\right| \leq 81B^2\epsilon$ and $|Q'(t_0)(\bar{t}_0 - t_0)| \leq 81B^2\epsilon + \frac{1}{2}\left|Q''(t_0)(\bar{t}_0 - t_0)^2\right|$. Since $|Q''(t_0)| \leq 2B^2$, we get:

$$|Q'(t_0)(\bar{t}_0 - t_0)| \leq 25B^2\epsilon^{\frac{1}{2}} + 81B^2\epsilon. \quad (31)$$

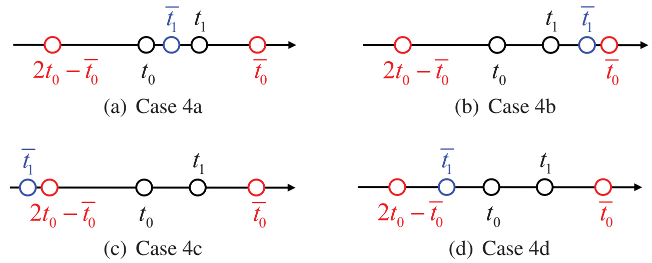
Using Equation 31 and Taylor expansion, for any $t \in [2t_0 - \bar{t}_0, t_0]$, we can formulate an upper bound on $|Q(t)|$ as:

$$\begin{aligned} |Q(t)| &\leq |-Q'(t_0)(t - t_0)| + \left|\frac{Q''(t_0)}{2}(t - t_0)^2\right| \\ &\leq |-Q'(t_0)(\bar{t}_0 - t_0)| + \left|\frac{Q''(t_0)}{2}(\bar{t}_0 - t_0)^2\right| \leq 51B^2\epsilon^{\frac{1}{2}}. \end{aligned} \quad (32)$$

Note that the above conclusions are valid for any candidate solved by Case 4, even when it is not a minimum of $G_{0ij}(t)$.

Our next goal is to study how $\bar{t}_0$ affects $G_{0ij}(\bar{t}_0)$. Without loss of generality, we assume that all of the Case 3 and Case 4 candidates are unique. If no candidate exists from t_0 to $\bar{t}_0$, we can formulate $G_{0ij}(t)$ from t_0 to $\bar{t}_0$ as: $G_{0ij}(t) = |Q(t)| + R(t)$, in which $R(t)$ is the residual quadratic component of $G_{0ij}(t)$. Since $G_{0ij}(t)$ is minimized at t_0, we must have $|Q'(t_0)| \geq |R'(t_0)|$. Therefore, $|R(\bar{t}_0) - R(t_0)| \leq \left|\frac{1}{2}R''(t_0)(\bar{t}_0 - t_0)^2\right| + |R'(t_0)(\bar{t}_0 - t_0)| \leq \left|\frac{1}{2}R''(t_0)(\bar{t}_0 - t_0)^2\right| + |Q'(t_0)(\bar{t}_0 - t_0)|$. Using the fact that $|R''(t_0)| \leq 4B^2$, we get $|R(\bar{t}_0) - R(t_0)| \leq 75B^2\epsilon^{\frac{1}{2}} + 81B^2\epsilon$. Combined with $|Q(\bar{t}_0) - Q(t_0)| \leq 81B^2\epsilon$, we know $\left|G_{0ij}(\bar{t}_0) - G_{0ij}(t_0)\right| \leq 76B^2\epsilon^{\frac{1}{2}}$. We note that for any $t \in [t_0, \bar{t}_0]$, we can find an intermediate function between $Q(t)$ and $\bar{Q}(t)$ whose exact solution is t. By treating t as $\bar{t}_0$ and the intermediate function as $\bar{Q}(t)$, we can get $\left|G_{0ij}(t) - G_{0ij}(t_0)\right| \leq 76B^2\epsilon^{\frac{1}{2}}$. It means $\left|G_{0ij}(\bar{t}_0) - G_{0ij}(t_0)\right| \leq 76B^2\epsilon^{\frac{1}{2}}$ is also valid after clamping $\bar{t}_0$ to $[0, 1]$.

If no collision candidate exists in $[2t_0 - \bar{t}_0, t_0]$, we would like to find the change of $G_{0ij}(t)$ in it as well. Since no candidate happens in $[2t_0 - \bar{t}_0, t_0]$, $G_{0ij}(t)$ is still $G_{0ij}(t) = |Q(t)| + R(t)$. Like before, we have: $|R(2t_0 - \bar{t}_0) - R(t_0)| \leq \left|\frac{1}{2}R''(t_0)(\bar{t}_0 - t_0)^2\right| + |R'(t_0)(\bar{t}_0 - t_0)| \leq 75B^2\epsilon^{\frac{1}{2}} + 81B^2\epsilon$. Combined with $|Q(2t_0 - \bar{t}_0) - Q(t_0)| \leq 51B^2\epsilon^{\frac{1}{2}}$ given in Equation 32, we know $\left|G_{0ij}(\bar{t}_0) - G_{0ij}(t_0)\right| \leq 127B^2\epsilon^{\frac{1}{2}}$. From the above analysis, it is straightforward to see that $\left|G_{0ij}(t) - G_{0ij}(t_0)\right| \leq 127B^2\epsilon^{\frac{1}{2}}$ is also true, for any $t \in [2t_0 - \bar{t}_0, t_0]$.



(a) Case 4a (b) Case 4b (c) Case 4c (d) Case 4d

Figure 10: *Different cases under Case 4. The existence of a new Case 4 collision time candidates between $2t_0 - \bar{t}_0$ and $\bar{t}_0$ will cause the error within the interval to increase.*

But what happens when there are collision time candidates between t_0 and $\bar{t}_0$, assuming that $t_0 \leq \bar{t}_0$? Here we consider the candidates belonging to Case 3 and 4 only, since they are the only ones that can change $G_{0ij}(t)$. To begin with, we first consider the change of $G_{0ij}(t)$ when a new Case 4 candidate t_1 is introduced. By definition, $G_{0ij}(t)$ is changed by $-2Q_1(t)$, when it passes through t_1 from t_0. Suppose that $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A$ for $t \in [2t_0 - \bar{t}_0, \bar{t}_0]$. There are five possible cases.

- **Case 4a.** In this case, $t_1 \in [t_0, \bar{t}_0]$ and $\bar{t}_1 \in [t_0, t_1]$, as Figure 10a shows. Since $G_{0ij}(t)$ was unchanged in $[2t_0 - \bar{t}_1, \bar{t}_1]$, we treat $\bar{t}_1$ as $\bar{t}_0$ and we still have: $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A$, for $t \in [2t_0 - \bar{t}_0, \bar{t}_0]$.

- **Case 4b.** If $t_1 \in [t_0, \bar{t}_0]$ and $\bar{t}_1 \geq t_1$ as Figure 10b shows, we set $\bar{t} = \min(\bar{t}_0, \bar{t}_1)$. Since $Q_1(\bar{t}) \leq 81B^2\epsilon$, we have: $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A + 2 \cdot 81B^2\epsilon$ for $t \in [2t_0 - \bar{t}, \bar{t}]$. So we treat $\bar{t}$ as $\bar{t}_0$.

- **Case 4c.** If $t_1 \in [t_0, \bar{t}_0]$ and $\bar{t}_1 \leq 2t_0 - \bar{t}_0$ as shown in Figure 10c, then $\bar{t}_0 \in [t_1, 2t_1 - \bar{t}_1]$ and we know $|Q_1(\bar{t}_0)| \leq 51B^2\epsilon^{\frac{1}{2}}$. Therefore, $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A + 2 \cdot 51B^2\epsilon^{\frac{1}{2}}$ for $t \in [2t_0 - \bar{t}_0, \bar{t}_0]$.

- **Case 4d.** If $t_1 \in [t_0, \bar{t}_0]$ and $\bar{t}_1 \in [2t_0 - \bar{t}_0, t_0]$ as shown in Figure 10d, then we treat $\bar{t}_1$ as $\bar{t}_0$. The function $G_{0ij}(t)$ is unchanged in $[\bar{t}_1, t_0]$, so we still have $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A$ for $t \in [\bar{t}_1, t_0]$. If $2t_0 - \bar{t}_1 \in [t_0, t_1]$, $G_{0ij}(t)$ is unchanged in $[t_0, 2t_0 - \bar{t}_1]$ as well. If not, since $2t_0 - \bar{t}_1 \leq 2t_1 - \bar{t}_1$, we have $|Q_1(2t_0 - \bar{t}_1)| \leq 51B^2\epsilon^{\frac{1}{2}}$, so $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A + 2 \cdot 51B^2\epsilon^{\frac{1}{2}}$ for $t \in [t_0, 2t_0 - \bar{t}_1]$. By treating $\bar{t}_1$ as $\bar{t}_0$, we know $|G_{0ij}(t) - G_{0ij}(t_0)| \leq A + 2 \cdot 51B^2\epsilon^{\frac{1}{2}}$ for $t \in [2t_0 - \bar{t}_0, \bar{t}_0]$.

- **Case 4e.** If $t_1 \in [2t_0 - \bar{t}_0, t_0]$, it also affects $G_{0ij}(t)$. Since we maintain the same error bound within the interval and the interval is symmetric to t_0, the analysis from Case 4a to 4d is also applicable here.

The above analysis indicates that when a new Case 4 candidate is introduced, the error bound within the constructed interval is increased by $102B^2\epsilon^{\frac{1}{2}}$ at most and one end of the interval is a computed candidate. Since there can be five additional Case 4 candidates, we must have: $|G_{0ij}(\bar{t}_0) - G_{0ij}(t_0)| \leq (127 + 5 \cdot 102)B^2\epsilon^{\frac{1}{2}} \leq 637B^2\epsilon^{\frac{1}{2}}$, for $t \in [2t_0 - \bar{t}_0, \bar{t}_0]$. We can then introduce Case 3 candidates and replace $\bar{t}_0$ by a Case 3 candidate, if it is within $[2t_0 - \bar{t}_0, \bar{t}_0]$. In the end, we still have $|G_{0ij}(\bar{t}_0) - G_{0ij}(t_0)| \leq 637B^2\epsilon^{\frac{1}{2}}$.

If $\bar{t}_0$ is given by Case 3, it is exact. According to Case 3, we have $|G_{0ij}(\hat{t}_0) - G_{0ij}(\bar{t}_0)| \leq B^2O(\epsilon)$. If $\bar{t}_0$ is given by Case 4, it is the exact solution provided by the quadratic solver. Appendix A.1 shows that $|\hat{t}_0 - \bar{t}_0| \leq 2\epsilon^{\frac{1}{2}} + O(\epsilon)$. So by definition, we get: $|G_{0ij}(\hat{t}_0) - G_{0ij}(\bar{t}_0)| \leq 6B^2|\hat{t}_0 + \bar{t}_0||\hat{t}_0 - \bar{t}_0| + 13B^2|\hat{t}_0 - \bar{t}_0| \leq 51B^2\epsilon^{\frac{1}{2}}$. So in both cases,we have

$|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)| \leq 688B^2\epsilon^{\frac{1}{2}}$. This conclusion is valid for any t in $[t_0, \hat{t}_0]$ as well.

In summary, we have $|G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)| \leq 688B^2\epsilon^{\frac{1}{2}}$ for all of the four cases. Since $G_{0ij}(t)$ is derived from $F_{0ij}(t)$ by removing the components of $\mathbf{x}_a$ and $\mathbf{x}_b$ when they are less than ρ and there are six components in total, we know $|G_{0ij}(t) - F_{0ij}(t)| \leq 6\rho$, for any $t \in [0, 1]$. One small problem is that ρ as $3B^2\epsilon^{\frac{1}{2}}$ cannot be exactly computed. To ensure that the computed $\hat{\rho}$ is greater than ρ, we can either use the `round-to-ceiling` scheme or multiply each intermediate result with a factor $1 + \epsilon$. By doing this, we can get $\hat{\rho} \geq \rho$ and $|\hat{\rho} - \rho| \leq B^2O(\epsilon)$. Let t_0^{F} be the exact time when $F_{ij}(t)$ gets minimized locally, we have:

$$G_{0ij}(t_0) - F_{0ij}(t_0^{\mathrm{F}}) \leq G_{0ij}(t_0^{\mathrm{F}}) - F_{0ij}(t_0^{\mathrm{F}}) \leq 18B^2\epsilon + B^2O(\epsilon). \quad (33)$$

So we have:

$$\begin{aligned} F_{0ij}(\hat{t}_0) - F_{0ij}(t_0^{\mathrm{F}}) \leq |F_{0ij}(\hat{t}_0) - G_{0ij}(\hat{t}_0)| + \\ |G_{0ij}(\hat{t}_0) - G_{0ij}(t_0)| + G_{0ij}(t_0) - F_{0ij}(t_0^{\mathrm{F}}) \leq 725B^2\epsilon^{\frac{1}{2}}. \end{aligned} \quad (34)$$

For any $t \in [t_0, \hat{t}_0]$, using the fact that $|G_{0ij}(t) - G_{0ij}(t_0)| \leq 688B^2\epsilon^{\frac{1}{2}}$, we can treat t as $\hat{t}_0$ and still get $F_{0ij}(t) - F_{0ij}(t_0^{\mathrm{F}}) \leq 725B^2\epsilon^{\frac{1}{2}}$. If $t \in [t_0^{\mathrm{F}}, t_0]$, we can treat t as t_0, which is supposed to be the local minimum of $G_{0ij}(t)$ closest to t_0^{F} (within $[t_0^{\mathrm{F}}, t_0]$). So we still have $F_{0ij}(t) - F_{0ij}(t_0^{\mathrm{F}}) \leq 725B^2\epsilon^{\frac{1}{2}}$. Together, we know $F_{0ij}(t) - F_{0ij}(t_0^{\mathrm{F}}) \leq 725B^2\epsilon^{\frac{1}{2}}$ is true for any $t \in [t_0^{\mathrm{F}}, \hat{t}_0]$.