Stable Discrete Bending by Analytic Eigensystem and Adaptive Orthotropic Geometric Stiffness

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In this paper, we address two limitations of dihedral angle based discrete bending (DAB) models, i.e. the indefiniteness of their energy Hessian and their vulnerability to geometry degeneracies. To tackle the indefiniteness issue, we present novel analytic expressions for the eigensystem of a DAB energy Hessian. Our expressions reveal that DAB models typically have positive, negative, and zero eigenvalues, with four of each, respectively. By using these expressions, we can efficiently project an indefinite DAB energy Hessian as positive semi-definite analytically. To enhance the stability of DAB models at degenerate geometries, we propose rectifying their indefinite geometric stiffness matrix by using orthotropic geometric stiffness matrices with adaptive parameters calculated from our analytic eigensystem. Among the twelve motion modes of a dihedral element, our resulting Hessian for DAB models retains only the desirable bending modes, compared to the undesirable altitude-changing modes of the exact Hessian with original geometric stiffness, all modes of the Gauss-Newton approximation without geometric stiffness, and no modes of the projected Hessians with inappropriate geometric stiffness. Additionally, we suggest adjusting the compression stiffness according to the Kirchhoff-Love thin plate theory to

Fig. 1. We illustrate the effectiveness of our approach with a T-shirt example. In the first row, we simulate the same T-shirt using five different scuba fabrics, each with the same high bending stiffness but different membrane stiffness. From (a) to (e), the lower the ratio of bending stiffness to membrane stiffness (BMR), the higher the membrane stiffness. In the second row, we use five different fabrics with the same small membrane stiffness but varying bending stiffness properties. From (f) to (j), the lower the BMR, the lower the bending stiffness. Note that figures (a) and (f) use the same scuba fabric with high bending stiffness and small membrane stiffness. Our proposed method allows for the stable simulation of fabrics with a wide range of BMRs. (BMR x1 corresponds to a Kirchhoff-Love thin plate with a thickness of 1 mm. Given a BMR xN, the corresponding thickness is \(\sqrt{N}\) mm. For example, BMR x16 corresponds to 4 mm.)
avoid over-compression. Our method not only ensures the positive semi-definiteness but also avoids instability caused by large bending forces at degenerate geometries. To demonstrate the benefit of our approaches, we show comparisons against existing methods on the simulation of cloth and thin plates in challenging examples.


Additional Key Words and Phrases: discrete bending, dihedral angle, eigen-system, geometric stiffness

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1 INTRODUCTION

Physics-based simulation of cloth and thin plates are widely used in industrial fashion design and animation movies to create realistic folds and wrinkles. The formation of these folds and wrinkles depends on constitutive models for bending deformation. While elasticity models have received a lot of attention in the graphics community and are constantly advancing, bending models have progressed slowly since their boom in the early 21st century.

Dihedral angle based discrete bending (DAB) models have been widely used in the simulation of cloth, thin plates, and thin shells. However, the two main limitations of using dihedral angles, as pointed out by Tamstorf and Grinspun [2013], have been overlooked for a long time. First, the Hessian of a dihedral angle, also the geometric stiffness matrix, is indefinite, as demonstrated in Section 4.1. This indefiniteness is due to the non-linearity and non-convexity of a dihedral angle with respect to positions. Second, geometry degeneracies, such as edge degeneration and altitude collapse, make the computation of bending forces and Hessian vulnerable to divergence, leading to instability of simulation, especially when bending stiffness dominates membrane stiffness and large bending forces can instantaneously over-stretch or over-compress triangles. Existing methods typically assume triangle meshes undergoing (near-) isometric deformation, where strong enough membrane stiffness can effectively act as geometric stiffness for DAB models. As a result, the impact of an indefinite bending energy Hessian on simulation stability is negligible [Grinspun et al. 2003; Tamstorf and Grinspun 2013], and approximations can be employed to simplify the computation of DAB models [Bergou et al. 2006b; Bridson et al. 2003]. Moreover, degenerate geometries are no longer problems in (near-) isometric simulations.

However, this assumption is not universally valid in industrial interactive applications, where users should possess the autonomy to adjust membrane and bending stiffness independently, thereby achieving desired simulation effects without being constrained by realistic material limitations. While simulation stability can be attained through nonlinear methods with small step lengths, in industrial contexts, it is often advantageous to employ a sole iteration of Newton’s method accompanied by a limited number of PCG iterations as the linear solver [Baraff and Witkin 1998; Tournier et al. 2015; Wu et al. 2022]. For instance, in real-time interactive fashion design, this strategy strikes a balance between interactive performance and robustness without expensive step-length line-searching. Therefore, the instability concerns for DAB models become significant, particularly when bending stiffness dominates membrane stiffness for special visual effects. While a high bending-membrane ratio (BMR) is uncommon in the realm of realistic continuum materials, it can readily manifest in specialized composite materials. As displayed in Fig. 1, certain scuba fabrics [Yip and Ng 2008; Zhang et al. 2020] exhibit high BMRs with fabric thickness up to 4 ~ 7.5 mm. To ensure both interactive robustness and user-friendliness, it is imperative to refrain from constraining the range of permissible BMRs and uphold the simulation stability across a broad range of BMRs.

Existing methods for restoring the positive semi-definiteness of FEM elastic energy Hessian [Smith et al. 2019; Teran et al. 2005] are not applicable to DAB models. Singular value decomposition (SVD) can be used to restore the positive semi-definiteness of an energy Hessian, but it is computationally expensive and suffers potential numerical instability. To address indefiniteness, we present novel analytic expressions for the eigensystem of DAB models. Our method reveals that DAB models typically have positive, negative, and zero eigenvalues, with four of each, when a dihedral element is away from its reference state. Therefore, negative eigenvalues can be clamped to zero using our analytic expressions, which is more efficient and stable than using SVD. Our eigenanalysis is based on a concise matrix expression for the dihedral angle Hessian, which serves as the geometric stiffness of a dihedral element. This expression reveals that only the undesirable altitude-changing modes of a dihedral element are retained under the effect of the geometric stiffness. Furthermore, we observe that the Hessian projection cannot promise stability in only one Newton’s iteration when the BMR is high, as it eliminates all motion modes of a dihedral element. To enhance stability at degenerate geometries, we propose applying orthotropic geometric stiffness with adaptive parameters. This approach retains only the desirable bending modes and effectively suppress instantaneous over-stretching and over-compression caused by large bending forces. Since we leave bending forces unmodified, over-compression is inevitable where bending forces dominate membrane forces. We notice that thin materials are difficult to compress in plane if buckling is prevented, regardless of their ease of stretching. Therefore, we suggest adjusting the local compression stiffness of each triangle element according to the Kirchhoff-Love thin plate theory to avoid degenerate geometries. Because it is an adaptive adjustment based on the strain of each triangle, the resulting compression stiffness maintains C1 continuity near the reference state, facilitating smooth transitions between in-plane compression and stretching. To be summarized, our main contributions include:

• we provide a concise derivation of the dihedral angle Hessian and present novel analytic expressions for the eigensystem of DAB models. This enables us to restore the positive semi-definiteness of the energy Hessian efficiently;
• we propose an orthotropic geometric stiffness model with parameters determined by the analytic eigensystems to enhance the simulation stability at degenerate geometries;
• we suggest adjusting the compression stiffness based on the Kirchhoff-Love thin plate theory to handle triangle over-compression and avoid geometry degeneracies.
Overall, our contributions improve the stability of DAB models and enable them to handle a wide range of materials in linearized simulation with only one Newton’s iteration without a line search, making them more useful for industrial interactive fashion design and animation movies.

2 RELATED WORK

In physics-based simulations, bending models are essential for accurately representing the appearance of thin materials modeled by discrete surfaces. These models typically rely on geometric curvatures that play important roles in both geometric and physical modeling [Grinspun et al. 2003; Sullivan 2005, 2008]. Meyer et al. [2003] presented a method to evaluate discrete mean curvature and Laplace-Beltrami for triangulated 2-manifolds. Based on restricted Delaunay triangulation and normal cycle, Cohen-Steiner and Morvan [2003] defined several discrete curvature measures. Different discrete elements lead to different curvature measures, including vertex-based and edge-based curvature measures [Polthier et al. 2002], as well as triangle-based shape operators [Gingold et al. 2004; Grinspun et al. 2006]. Related reviews can be found in [Grinspun 2006; Wacker and Morvan 2013; Sullivan 2005, 2008]. Meyer et al. [2007; Chen et al. 2018] are typically required. Alternatively, one can use piecewise linear finite elements and model bending energy on dihedral elements. In this paper, we specifically focus on the discrete bending model defined on linear finite elements, which is widely used in physics-based simulation. For a discrete dihedral element, the bending energy is built on dihedral angle based discrete edge-based curvature. Baraff and Witkin [1998] introduced a bending constraint that restricts the dihedral angle of an edge to model the bending behaviors of clothing. Grinspun et al. [2003] defined the bending energy of discrete shells from the perspective of mean curvature difference, based on the conclusion of [Cohen-Steiner and Morvan 2003]. Bridson et al. [2003] directly generated bending forces for a dihedral element, with a magnitude of the cosine of half the dihedral angle and well-designed directions that are actually the gradient of a dihedral angle. They also imposed explicit damping forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability. To embed bending energy in the framework of implicit time integration, the complicated gradient forces to enhance stability.

3 BACKGROUND

Before diving into the investigation of existing and our approaches for stabilizing discrete bending models, we would like to provide a brief introduction to their background. In this paper, we use unbolded characters in lowercase (a) or uppercase (A) to denote scalars, bolded lowercase (a) for column vectors, and bolded uppercase (A) for matrices. A bar means a reference value. ⊗ represents the Kronecker product. The definition of frequently used symbols can be found in Table. 1.

3.1 DAB Models

A dihedral element is composed of four vertices \( x_i \in \mathbb{R}^3 (i \in \{0, 1, 2, 3\}) \) and two adjacent triangles that share a common hinge edge, as shown in Fig. 2(a). Its reference area is \( \bar{A} / 3 \), which is one-third of the reference triangle area \( \bar{A} \), and the rest length of the hinge edge is \( \bar{l} \). A general bending energy on the element can be defined w.r.t. the packed position \( \bar{x} = [x_0^T, x_1^T, x_2^T, x_3^T]^T \in \mathbb{R}^{12} \) as

\[
\Psi(\bar{x}) = \mu \psi(\bar{\theta}, \bar{\bar{\theta}}),
\]

where \( \mu \) is a weight, \( \psi(\bar{\theta}, \bar{\bar{\theta}}) \) is a function of the dihedral angle \( \bar{\theta} \) and its second derivative \( \bar{\bar{\theta}} \). Experiments and theoretical analysis suggest that for a bending model with \( \Psi(\bar{x}) \), the linearized energy will be independent of \( \mu \).

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where $\theta$ and $\bar{\theta}$ are the deformed and reference dihedral angles, and $\mu = 3F/\bar{A}$ is a reference shape factor. Different choices of $\psi(\bar{\theta}, \bar{\theta})$ result in several bending models:

- $\psi(\bar{\theta}, \bar{\theta}) = \frac{1}{2}(\bar{\theta} - \bar{\theta})^2$: referred to as Discrete Shells [Grinspun et al. 2003];
- $\psi(\bar{\theta}, \bar{\theta}) = 2(\sin \bar{\theta} - \bar{\theta})^2$: referred to as Cubic Shells [Garg et al. 2007] whose energy is cubic for unstretched surfaces;
- $\psi(\bar{\theta}, \bar{\theta}) = 2(\cos \bar{\theta})^2$: referred to as Discrete Willmore Energy [Wardetzky et al. 2007]. It is equivalent to Quadratic Bending [Bergou et al. 2006b] for inextensible surfaces;
- $\psi(\bar{\theta}, \bar{\theta}) = \cos \bar{\theta} - 2 \sin \bar{\theta}/2$: proposed by Bridson et al. [2003] for the simulation of clothing with folds and wrinkles and referred to as Bridson’s model in this paper;
- $\psi(\bar{\theta}, \bar{\theta}) = 2(\cot \bar{\theta} - \cot \bar{\theta})^2$: proposed by Tamstorf and Grinspun [2013] to avoid penetrations and referred to as Tamstorf’s model in this paper.

Recently, some studies [Romero et al. 2021] have emphasized the importance of considering $\theta - \bar{\theta}$ rather than $\theta$ or $\bar{\theta}$ individually for simulation results being consistent with physical reality. To integrate these models into implicit time integration, expressions of bending force and energy Hessian are required,

$$f = -\mu g \nabla x \vartheta \quad \text{and} \quad H = \mu (pP + gG),$$

in which $g = \frac{\partial \psi}{\partial \vartheta}$ and $p = \frac{\partial^2 \psi}{\partial \vartheta^2}$, $\nabla x \vartheta \in \mathbb{R}^{12}$ (Eq. 3) is the dihedral angle gradient, $G = -\frac{\partial^2 \psi}{\partial \vartheta^2} e_{12} \in \mathbb{R}^{12 \times 12}$ (Eq. 4) is the dihedral angle Hessian and $P = \frac{\partial \vartheta}{\partial x} \frac{\partial \vartheta}{\partial x}^T \in \mathbb{R}^{12 \times 12}$ is a positive semi-definite projection matrix that projects a vector onto the subspace spanned by the dihedral angle gradient. Tamstorf and Grinspun [2013] conducted a thorough analysis of the calculation of these terms. However, their derivation and Hessian expressions are cumbersome. To facilitate the analysis of the instability of DAB models, we present a more concise matrix expression for the Hessian of a dihedral angle in Section 4.1.

### 3.2 Problems

Tamstorf and Grinspun [2013] identified two primary limitations of DAB models: the indefiniteness of a dihedral angle Hessian and the vulnerability to geometry degeneracies.

#### 3.2.1 Indefiniteness

If a dihedral element undergoes bending deformation, the bending forces and dihedral angle Hessian $G$ will come into play due to $g \neq 0$. The presence of $G$, whose indefiniteness will be demonstrated in Section 4.1, may result in negative eigenvalues for the energy Hessian $H$. In some DAB models, such as Cubic Shells, Discrete Willmore Energy and Tamstorf’s model, negative eigenvalues can arise from a negative $p$ when the dihedral angle difference is large. Our analytic eigensystem in Section 4.2 demonstrates that $H$ is indefinite when a dihedral element is away from its reference bending state. Using an indefinite bending Hessian in cloth or thin plate simulation could lead to instability at large bending deformations, as illustrated in Fig. 5(a).

The most common method to address the indefiniteness of an energy Hessian is to perform Hessian Projection, which involves clamping the negative eigenvalue to zero. Exact Hessian projection can only be performed numerically using SVD, which is not computationally efficient. More efficient methods, such as [Smith et al. 2019; Teran et al. 2005], use the deformation gradient of finite elements to obtain a block-diagonal intrinsic deformation matrix that can be efficiently projected as positive semi-definite. However, they cannot be applied to DAB models because it is impossible to define a deformation-gradient-like variable for dihedral elements.

#### 3.2.2 Geometry Degeneracies

The computation of bending force and energy Hessian in DAB models involves dividing hinge edge length and altitudes, which can be problematic when edges become degenerate or altitudes collapse. Although divisions by zero can be numerically avoided, degenerate altitudes will produce large out-of-plane bending forces which can stretch or compress triangles excessively if the bending stiffness dominates the membrane stiffness. As illustrated in Fig. 8 and Fig. 10, geometry degeneracies occur at the corners where triangles are over-compressed by large bending forces. However, DAB models do not respond to in-plane

### Table 1. Frequently used symbols.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$\theta$</td>
<td>dihedral angle</td>
</tr>
<tr>
<td>$l$</td>
<td>hinge edge length</td>
</tr>
<tr>
<td>$h_1, h_2$</td>
<td>hinge altitudes</td>
</tr>
<tr>
<td>$\omega_1, \omega_2$</td>
<td>barycentric coords</td>
</tr>
<tr>
<td>$\psi$</td>
<td>energy density</td>
</tr>
<tr>
<td>$h$</td>
<td>thickness</td>
</tr>
<tr>
<td>$g$</td>
<td>$\frac{\partial \psi}{\partial \vartheta}$</td>
</tr>
<tr>
<td>$p$</td>
<td>$\frac{\partial^2 \psi}{\partial \vartheta^2}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$3l/A$</td>
</tr>
<tr>
<td>$A$</td>
<td>area sum</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>damping params</td>
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<tr>
<td>$Y$</td>
<td>Young’s modulus</td>
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<tr>
<td>$\mu_1, \mu_2$</td>
<td>triangle normals</td>
</tr>
<tr>
<td>$m_1, m_2$</td>
<td>altitude vectors</td>
</tr>
<tr>
<td>$e$</td>
<td>hinge edge vector</td>
</tr>
<tr>
<td>$h_1, h_2$</td>
<td>hinge edge length</td>
</tr>
<tr>
<td>$\omega_1, \omega_2$</td>
<td>barycentric coords</td>
</tr>
<tr>
<td>$y_1, y_2, y_3$</td>
<td>$\sum s[i] \Delta x_i$</td>
</tr>
<tr>
<td>$f$</td>
<td>bending force</td>
</tr>
<tr>
<td>$H$</td>
<td>energy Hessian</td>
</tr>
<tr>
<td>$\nabla \vartheta$</td>
<td>$\nabla x \vartheta$'s projection matrix</td>
</tr>
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Both Soft

We propose an adaptive orthotropic geometric stiffness model to compute the dihedral angle Hessian and a profound analysis for the geometric stiffness of DAB models. Second, we present an analytic eigensystem derived in [Tamstorf and Grinspun 2013]. To compute the dihedral angle Hessian, we may notice that the shape of a dihedral element is also influenced by deformations, which can affect the magnitude and direction of bending forces but have no influence on bending energies. While it may be acceptable to assume constant in-plane variables, e.g. $h_1, h_2, \omega_1, \omega_2$ and $l$, in an isometric simulation, doing so in a non-isometric simulation can lead to artifacts. In contrast, using constant in-plane variables produces an artifact at the seam in the second row of column (a).

4 OUR METHOD

To overcome the limitations of DAB models, we propose several novel techniques. First, we provide a compact matrix expression for the dihedral angle Hessian and a profound analysis for the geometric stiffness of DAB models. Second, we present an analytic eigensystem for the energy Hessian to address the indefiniteness issue. Third, we propose an adaptive orthotropic geometric stiffness model to improve the stability of DAB models. Finally, we suggest adjusting compression stiffness based on the Kirchhoff-Love plate theory to handle over-compression.

4.1 Dihedral Angle Hessian

In addition to the dihedral angle $\theta$ and hinge edge length $l$, we notice that the shape of a dihedral element is also influenced by four other pivotal variables: $h_1$ and $h_2$, altitudes on the hinge edge, and $\omega_1$ and $\omega_2$, barycentric weights of $x_2$ and $x_3$ on the hinge edge. By defining two vectors $t_1 = h_1^{-1} [\omega_1 - 1, -\omega_1, 1, 0]^T$ and $t_2 = h_2^{-1} [\omega_2 - 1, -\omega_2, 0, 1]^T$, the gradient of a dihedral angle can be expressed as a column vector in $\mathbb{R}^{12}$,

$$\nabla_x \theta = t_1 \otimes n_1 + t_2 \otimes n_2,$$

where $n_1$ and $n_2$ are unit normals of the two adjacent triangles, as shown in Fig. 2(b). Their gradients, $\nabla_x n_1$ and $\nabla_x n_2$, have been derived in [Tamstorf and Grinspun 2013]. To compute the dihedral angle Hessian $G$, we also need $\nabla_x h_1, \nabla_x h_2, \nabla_x \omega_1$ and $\nabla_x \omega_2$ to compute $\nabla_x t_1$ and $\nabla_x t_2$. The derivation of these gradients has been presented in detail in Appendix A.

4.1.1 Matrix expression. For a dihedral element depicted in Fig. 2, let $e$ be the normalized direction of the hinge edge, $m_1$ and $m_2$ be normalized altitude vectors. To understand the indefiniteness of the dihedral angle Hessian $G$, we provide its compact matrix expression,

$$G = G_m + G_c,$$

$$G_m = (t_1 t_1^T) \otimes A_1 + (t_2 t_2^T) \otimes A_2 + \frac{1}{2} \left(s s^T\right) \otimes \left(A_1 + A_2\right),$$

$$G_c = B_1 + B_1^T + B_2 + B_2^T,$$

where $s = l^{-1} [1, -1, 0, 0]^T$, $A_1 = m_1 n_1^T + n_1 m_1^T$ and $A_2 = m_2 n_2^T + n_2 m_2^T$ are matrices in $\mathbb{R}^{3 \times 3}$, $B_1 = (t_1 s^T) \otimes (e n_1^T)$ and $B_2 = (t_2 s^T) \otimes (e n_2^T)$ are matrices in $\mathbb{R}^{12 \times 12}$. $G$ is indefinite because of the presence of indefinite matrices $A_1, A_2, B_1 + B_1^T$ and $B_2 + B_2^T$ which have both positive and negative eigenvalues. For more details on our derivation, please refer to Appendix A. In spite of the indefiniteness of $G$, we cannot assert with certainty that the energy Hessian $H$ is indefinite due to the presence of $P$. In Section 4.2, our analytic eigensystem is the first to reveal that $H$ typically has four positive and four negative eigenvalues, in addition to four known zero eigenvalues.

As opposed to using complex Hessian projection techniques, a simpler approach to eliminate the indefiniteness issue is to omit $G$ and approximate the Hessian as $H = \mu P$. However, this approach may not guarantee positive semi-definiteness for some models that could result in a negative $\mu$. On the other hand, the Gauss-Newton method also omits $G$ but ensures the positive semi-definiteness for models with a quadratic energy $\psi(\theta, t) = \frac{1}{2} c^2(\theta, t)$ by using the approximation $H \approx \mu (\partial \psi / \partial \theta)P$, except for Bridson’s model.

4.1.2 Geometric stiffness. However, $G$ serves as geometric stiffness that controls the direction of displacements caused by bending force and omitting it can result in instability in some cases. The importance of the geometric stiffness have been demonstrated by Tournier et al. [2015] for a mass-spring system to suppress oscillations, and Choi and Ko [2002] suggested omitting the negative definite geometric stiffness only when a spring is compressed. In Section 4.2.1, we have described the twelve motion modes of a dihedral element in detail. We are interested in which displacement modes will be preserved under the effect of $H$. When solving a displacement equation $H \Delta x = f$ to find a Newton’s direction, $P$ only restricts the projection length of $\Delta x$ on the bending force $f$ and allows all the motion modes. To reveal the effect of geometric stiffness, we need to analyze the following two expressions,

$$G_m \Delta x = t_1 \otimes (A_1 y_1) + t_2 \otimes (A_2 y_2) + s \otimes (A_1 y_3 + A_2 y_3),$$

$$G_c \Delta x = (t_1^T y_3) t_1 \otimes e + (t_2^T y_3) t_2 \otimes e + (e^T y_1) s \otimes n_1 + (e^T y_2) s \otimes n_2,$$

where $y_1 = \sum t_1[i] \Delta x_i$ and $y_2 = \sum t_2[i] \Delta x_i$ are relative displacements on the two adjacent triangles, respectively, and $y_3 = \sum s[i] \Delta x_i$ on the hinge edge. Because $e$ is not contained in $f$, it follows that $n_1^T y_3 = 0$ and $n_2^T y_3 = 0$, implying that $y_3$ is parallel to $e$. Consequently, we have $A_1 y_3 + A_2 y_3 = 0$ resulting in the disappearance of $s$ in $G_m \Delta x$. Because $s$ is independent on $t_1$ and $t_2$, $s \otimes n_1$ and $s \otimes n_2$ are not contained in $f$, which also necessitates the elimination of $s$ in $G_c \Delta x$, leading to $e^T y_1 = 0$ and $e^T y_2 = 0$. In addition, $A_1 y_1$ should not contain $m_1$ and $A_2 y_2$ should not contain $m_2$. Therefore, we conclude that $y_1$ is parallel to $m_1$ and $y_2$ is parallel to $m_2$. Putting aside zero-eigenvalue motion modes, the remaining motion modes for a dihedral element under the effect of $H$ are altitude changes, i.e. $\Delta x = [-t_1^T, t_1^T, h_1 m_1^T, h_2 m_2^T]^T$, as depicted in Fig. 4(b). This implies that the indefinite energy Hessian $H$ tends to over-stretch the
dihedral element along the altitude directions. When using only one Newton’s iteration without a line search, no mechanism is in place to mitigate this detrimental consequence. As illustrated by Fig. 5(a), the right side of the cylinder shell undesirably expands during the twisting process. In scenarios involving low BMR and small bending deformation, a sufficiently strong membrane stiffness can alleviate the negative impact of an inappropriate geometric stiffness of DAB models. However, this negative effect can be more pronounced in the cases of high BMR and large bending deformation, where the geometric stiffness plays a dominant role. As illustrated in Fig. 7(g), Fig. 8 and Fig. 10(a, b), the Gauss-Newton approximation, lacking geometric stiffness, exhibits instability with oscillations. Additionally, as shown in Fig. 7(e, f), Hessian projections incorporating inappropriate geometric stiffness fail to stably simulate the compressing cylinder.

4.2 Analytic Eigensystem

We note that performing a full eigenanalysis of DAB models can provide a complete understanding of the nature of the energy Hessian and its eigenvalues. This information is crucial for determining the positive semi-definiteness or indefiniteness of H and addressing the indefiniteness issue. While previous works, such as [Bridson et al. 2003] and [Tamstorf and Grinspun 2013], have described the twelve distinct motion modes of a dihedral element and identified four zero eigenvalues, a full eigenanalysis has not been conducted until this work. By performing this analysis, we can gain insight into the behaviors of the model and potentially improve its performance.

4.2.1 The twelve modes. At the beginning of our eigenanalysis, we were faced with uncertainty. Fortunately, the twelve motion modes described by [Bridson et al. 2003] provide us a valuable guidance. These modes consist of three translations and three rotations representing rigid body motions, as well as two in-plane displacements for vertex $x_2$ and vertex $x_3$, the in-line stretching of the hinge edge, and a bending mode. To aid in our eigenanalysis for DAB models, we construct twelve displacement vectors that represent each of these motion modes explicitly.

Zero-eigenvalue modes. Constructing four zero-eigenvalue eigenvectors, namely $v_0$, $v_1$, $v_2$, and $v_3$ in Eq. 5, it is easy to verify that all of them satisfy $HV_j = 0$, and they correspond to three rigid translations and the uniform scaling along the hinge edge, respectively.

Nonzero-eigenvalue modes. For nonzero eigenvalues, there are eight motion modes or their combinations. As depicted in Fig. 4(a), barycentric sliding motions along the hinge edge of vertex $x_2$ and $x_3$ can be defined as $v_4$ and $v_5$, respectively. They do not affect the bending energy but alter the distribution of bending forces on the hinge edge. Similarly, we can define $v_6$ and $v_7$ for altitude-changing motions of vertex $x_2$ and $x_3$, respectively, as depicted in Fig. 4(b). They do not affect the bending energy but change the magnitude of bending forces. The bending modes correspond to dihedral angle changes without introducing in-plane deformation and rigid transformation, which can be represented by rotating vertex $x_2$ and $x_3$ around the hinge edge respectively. Therefore, we can define $v_8$ and $v_9$ for instance bending displacements of vertex $x_2$ and $x_3$, respectively, as depicted in Fig. 4(c). The only remaining modes are the three rigid rotations. The rotation around $e$ is linearly relative to the two bending modes. Therefore, only rotations around $n_1$ and $m_1$ need to be considered. The displacements of vertex $x_2$ and $x_3$ can be ignored because they are linearly relative to the sliding motions and altitude changes when rigid rotations occur. Assuming the midpoint of the hinge edge is the rotation origin, we can define $v_{10}$ and $v_{11}$ for rotation around $m_1$ and $n_1$ respectively.

\[ v_0 = [1, 1, 1]^T \otimes n_1, \quad v_1 = [1, 1, 1]^T \otimes m_1, \]
\[ v_2 = [1, 1, 1]^T \otimes e, \quad v_3 = [0, 1, 0, 0]^T \otimes e, \]
\[ v_4 = [0, 0, 1, 0]^T \otimes e, \quad v_5 = [0, 0, 1, 0]^T \otimes e, \]
\[ v_6 = [0, 0, 1, 0]^T \otimes m_1, \quad v_7 = [0, 0, 1, 0]^T \otimes m_2, \]
\[ v_8 = [0, 0, 1, 0]^T \otimes n_1, \quad v_9 = [0, 0, 1, 0]^T \otimes n_2, \]
\[ v_{10} = [1, -1, 0, 0]^T \otimes n_1, \quad v_{11} = [1, -1, 0, 0]^T \otimes m_1. \]

Although the eight vectors $v_j$ are orthogonal to each other, they are not eigenvectors of H because they do not satisfy $HV_j = \lambda_j v_j$ and are not orthogonal to the zero-eigenvalue eigenvectors $v_j(i \in \{0,...,3\})$. However, the expressions of $HV_j$ (presented in detail in Appendix B.1) guide us to find an invariant subspace of H, based on which we can discover an intrinsic decomposition of H.

4.2.2 Intrinsic decomposition. Directly analyzing the eigensystem of the energy Hessian H is a challenging task. However, inspired by the methods of [Smith et al. 2019; Teran et al. 2005], which cannot be
4.2.3 Analytic eigenvalues and eigenvectors. By exploiting the symbolic calculation capabilities of Mathematica to solve the characteristic polynomials of $F_0$ and $F_1$, we successfully obtain the analytic eigenvalues of $F$, i.e.

$$
\begin{align*}
\lambda_0 &= p + \sqrt{p^2 + g^2}, & \lambda_4 &= \frac{g}{2} \left( \sin \theta + \sqrt{\sin^2 \theta + 4(1 - \cos \theta)} \right), \\
\lambda_1 &= g, & \lambda_5 &= \frac{g}{2} \left( -\sin \theta + \sqrt{\sin^2 \theta + 4(1 + \cos \theta)} \right), \\
\lambda_2 &= p - \sqrt{p^2 + g^2}, & \lambda_6 &= \frac{g}{2} \left( \sin \theta - \sqrt{\sin^2 \theta + 4(1 - \cos \theta)} \right), \\
\lambda_3 &= -g, & \lambda_7 &= \frac{g}{2} \left( -\sin \theta - \sqrt{\sin^2 \theta + 4(1 + \cos \theta)} \right),
\end{align*}
$$

in which $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ are eigenvalues of $F_0$ and $F_1$, respectively. The eigenvalues of $F_0$ for different models have been depicted in Fig. 6(a-e). Only Tamstorf’s model has unbounded $F_0$ eigenvalues due to its unbounded bending energy. The eigenvalues of $F_1 / g (g \neq 0)$ depicted in Fig. 6(f) are the same for all these models, which reveals that additional zero eigenvalues appear at $\theta = 0, \pi, 2\pi$.

If a dihedral element has no bending deformation with $g = 0$, $F$ has only one nonzero eigenvalue $\lambda = 2p$. The corresponding eigenvector is $e = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, -\cos \theta, \sin \theta, 0, 0, 0, 0, 0 \end{bmatrix}^T$ and $Ze$ is the eigenvector of $H$ because of $HZ = \muZFZ^T Z$, where $F$ is only related to intrinsic bending deformation of a dihedral element. The detailed derivation of $F$ can be found in Appendix B.2. Because $Z$ has a full column rank, we obtain an important decomposition of the energy Hessian of DAB models, i.e.

$$
H = \muZFZ^T.
$$

Interestingly, $F$ is a block-diagonal matrix thanks to the $\zeta$'s in $Z$. Specifically, the upper-left block is

$$
F_0 = \begin{pmatrix}
p & -p \cos \theta & p \sin \theta & g \\
-p \cos \theta & p \cos^2 \theta - g \sin 2\theta & -p \sin^2 \theta + g \cos 2\theta & 0 \\
p \sin \theta & -p \sin^2 \theta - g \cos 2\theta & p \sin^2 \theta + g \sin 2\theta & 0 \\
g & 0 & 0 & 0
\end{pmatrix},
$$

and the bottom-right block is

$$
F_1 = g \begin{pmatrix}
-\sin \theta & \sin^2 \theta & -\cos \theta & 1 \\
\sin^2 \theta & \sin \theta & \cos \theta & 0 \\
-\cos \theta & \sin \theta & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

Notably, $F$ depends only on dihedral angles because $g$ and $p$ are only dependent on $\theta$ and $\delta$. 

4.2.3 Analytic eigenvalues and eigenvectors. By exploiting the symbolic calculation capabilities of Mathematica to solve the characteristic polynomials of $F_0$ and $F_1$, we successfully obtain the analytic eigenvalues of $F$, i.e.

$$
\begin{align*}
\lambda_0 &= p + \sqrt{p^2 + g^2}, & \lambda_4 &= \frac{g}{2} \left( \sin \theta + \sqrt{\sin^2 \theta + 4(1 - \cos \theta)} \right), \\
\lambda_1 &= g, & \lambda_5 &= \frac{g}{2} \left( -\sin \theta + \sqrt{\sin^2 \theta + 4(1 + \cos \theta)} \right), \\
\lambda_2 &= p - \sqrt{p^2 + g^2}, & \lambda_6 &= \frac{g}{2} \left( \sin \theta - \sqrt{\sin^2 \theta + 4(1 - \cos \theta)} \right), \\
\lambda_3 &= -g, & \lambda_7 &= \frac{g}{2} \left( -\sin \theta - \sqrt{\sin^2 \theta + 4(1 + \cos \theta)} \right),
\end{align*}
$$

in which $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ are eigenvalues of $F_0$ and $F_1$, respectively. The eigenvalues of $F_0$ for different models have been depicted in Fig. 6(a-e). Only Tamstorf’s model has unbounded $F_0$ eigenvalues due to its unbounded bending energy. The eigenvalues of $F_1 / g (g \neq 0)$ depicted in Fig. 6(f) are the same for all these models, which reveals that additional zero eigenvalues appear at $\theta = 0, \pi, 2\pi$.

If a dihedral element has no bending deformation with $g = 0$, $F$ has only one nonzero eigenvalue $\lambda = 2p$. The corresponding eigenvector is $e = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, -\cos \theta, \sin \theta, 0, 0, 0, 0, 0 \end{bmatrix}^T$ and $Ze$ is the eigenvector of $H$ because of $HZ = \muZFZ^T Z$, Otherwise in the case of $g > 0 (g < 0)$, $F$ has four positive eigenvalues $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ and four negative eigenvalues $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$, respectively. The corresponding eigenvectors of $F_0$ are

$$
\begin{align*}
e_0 &= \hat{e}(\lambda_0, 1), & \hat{e}_1 &= \hat{e}(\lambda_1, -1), & \hat{e}_2 &= \hat{e}(\lambda_2, 1), & \hat{e}_3 &= \hat{e}(\lambda_3, -1), \\
e_4 &= \hat{e}(\lambda_4, 1), & \hat{e}_5 &= \hat{e}(\lambda_5, -1), & \hat{e}_6 &= \hat{e}(\lambda_6, 1), & \hat{e}_7 &= \hat{e}(\lambda_7, -1),
\end{align*}
$$

in which

$$
\begin{align*}
\hat{e}(\lambda, \delta) &= \delta \left( \sin \theta - \cos \theta \frac{\delta}{g} \right), & \hat{e}(\cos \theta + \sin \theta \frac{\delta}{g}) & = \sin \theta \left( \cos \theta + \sin \theta \frac{\delta}{g} \right), \\
\hat{e}(\lambda, \delta) &= \delta \left( \sin \theta - \cos \theta \frac{\delta}{g} \right), & \hat{e}(\cos \theta + \sin \theta \frac{\delta}{g}) & = \sin \theta \left( \cos \theta + \sin \theta \frac{\delta}{g} \right),
\end{align*}
$$

with $\delta = \pm 1$ as a sign indicator, are the general expressions of eigenvectors of $F_0$ and $F_1$, respectively. One thing to note is that $\hat{e}(\lambda, \delta)$ is not continuous in some special cases where $F_1$ has zero eigenvalues:

- If $\theta = 0$ or $\theta = 2\pi$, there are $\lambda_4 = 0$ and $\lambda_5 = 0$. As a result, we can choose $\hat{e}_4 = [0, 1 + \sqrt{3}, 1, 1]^T$ and $\hat{e}_5 = [0, 1 - \sqrt{3}, 1, 1]^T$.
- If $\theta = \pi$, there are $\lambda_2 = 0$ and $\lambda_6 = 0$. As a result, we can choose $\hat{e}_2 = [0, 1 + \sqrt{3}, -1, 1]^T$ and $\hat{e}_6 = [0, 1 - \sqrt{3}, -1, 1]^T$. Note: $1 + \sqrt{3}$ and $1 - \sqrt{3}$ are the left and right limits of $\frac{\sin \theta}{\sqrt{\cos \theta}}$, respectively.
Consequently, we obtain the analytic eigendecomposition of the intrinsic bending matrix, i.e.

\[ F = E A E^T, \]  

(10)

where \( E \in \mathbb{R}^{8 \times 8} \) is an orthogonal matrix with columns as the normalized eigenvectors of \( F \), and \( A = \text{diag}(\lambda_0, \ldots, \lambda_7) \in \mathbb{R}^{8 \times 8} \) is a diagonal matrix. Combining Eq. 7 and Eq. 10 yields a new decomposition of the energy Hessian, i.e. \( H = \mu (Z E ) A (Z E )^T \). However, because of the non-orthogonality of matrix \( Z E \), the eigenvalues of \( F \) are not the eigenvalues of \( H \). Nonetheless, we prove based on Sylvester’s law of inertia that the numbers of positive, negative, and zero eigenvalues of \( H \) are the same as those of \( F \), which means that \( H \) also typically has four positive eigenvalues and four negative eigenvalues, as well as four zero eigenvalues. For more information on our proof, please refer to Appendix B.2.

4.2.4 F-projection. Although the intrinsic decomposition in Eq. 7 is not a similarity transform and can modify the spectrum, the positive semi-definiteness of \( F \) guarantees the positive-semi definiteness of \( H \). In order to restore the positive semi-definiteness of \( H \), we can perform direct modifications of \( H \), referred to as \( H \)-projection, or direct modifications of \( F \), referred to as \( F \)-projection. However, as far as we know, the \( H \)-projection can only be performed numerically using the SVD. Using the analytic eigensystem of \( F \) in Eq. 10, we can restore the positive semi-definiteness of \( H \) by setting negative eigenvalues of \( F \) to zero or a small positive value, such as \( 1e^{-6} \). We have summarized our \( F \)-projection algorithm for DAB models in Algorithm 1, which is more than 30 times faster than the SVD-based \( H \)-projection implemented using the eigenvalue() and eigenvectors() interfaces from Eigen library on the CPU.

As illustrated by the examples of twisting in Fig. 5 and the compressing example in Fig. 17, our \( F \)-projection produces stable simulations. However, the \( F \)-projection fails to stabilize the simulation of a cylinder plate with a high bending-membrane ratio in the compressing example in Fig. 7. To understand this failure, we consider the case of high BMR where bending force and Hessian dominate. This is supported by the low convergence of \( F \)-projection in Fig. 12. Although we cannot explicitly analyze the \( H \)-projection, its low convergence suggests that it also excludes all the motion modes.

4.3 Adaptive Orthotropic Geometric Stiffness (AOGS)

The exact geometric stiffness matrix \( G \) is indefinite and retains only the altitude-changing modes, which are undesirable for dihedral elements. While \( H \)-projection and \( F \)-projection are positive semi-definite, their geometric stiffness eliminate all motion modes and result in unstable simulations. On the other hand, the Gauss-Newton method is positive semi-definite but permits all motion modes, which is too permissive. The Levenberg-Marquardt algorithm (LMA) adds a damping Hessian into the Gaussian-Newton method, but it only dampens the bending modes and has no effect on other modes. Besides zero-eigenvalue modes, the ideal geometric stiffness for DAB models must be positive semi-definite and should only keep the bending modes and eliminate other non-bending modes.
which is a combination of stiffness matrix \( P = q_0 q_1^T \) and seven directional geometric stiffness terms \( q_i q_i^T \). To verify the effectiveness of these directions, we need to analyze the solution of the displacement equation \( H_q \Delta x = f \). Because only \( q_0 \) is parallel to the bending force \( f \), displacements along other directions should be eliminated (or damped), implying \( (q_i q_i^T) \Delta x = 0 \) for \( i \neq 0 \). Considering \( q_i^T \Delta x = 0 \) and \( q_0^T \Delta x = 0 \), we obtain two constraints for the target displacement,

\[
\begin{align*}
    m_1^T y_1 + m_2^T y_2 &= 0, & e^T y_1 + e^T y_2 &= 0,
\end{align*}
\]

in which \( y_{1,2} \) are relative displacements (Table 1). Considering \( q_i^T \Delta x = 0 \) and \( q_0^T \Delta x = 0 \), we obtain three constraints,

\[
\begin{align*}
    n_1^T y_1 - n_2^T y_2 &= 0, & n_1^T y_1 - m_2^T y_2 &= 0, & e^T y_1 - e^T y_2 &= 0.
\end{align*}
\]

By combining all the five constraints, we conclude that \( y_1 \) is parallel to \( n_1, y_2 \) is parallel to \( n_2 \) and \( \|y_1\| = \|y_2\| \). Excluding rigid translations, we obtain the expression of the target displacement,

\[
\Delta x = -\frac{1}{\alpha_0} \left( \{0^T, 0^T, h_1 n_1^T, h_2 n_2^T\}^T + \{0, 1, \omega_1, \omega_2\}^T \otimes y_3 \right),
\]

including the bending modes and uniform scaling along arbitrary directions. To further eliminate undesirable (uniform) scaling orthogonal to \( e \), we consider \( q_0^T \Delta x = 0 \) and \( q_0^T \Delta x = 0 \) to get another two constraints,

\[
\begin{align*}
    n_1^T y_3 + n_2^T y_3 &= 0, & n_1^T y_3 - n_2^T y_3 &= 0,
\end{align*}
\]

for relative displacement on the hinge edge \( y_3 \) being parallel to \( e \). The final displacement with zero-ellipse motion modes excluded is \( \Delta x = -\frac{1}{\alpha_0} \left( \{0^T, 0^T, h_1 n_1^T, h_2 n_2^T\}^T \right) \), as depicted in Fig. 4(c). These results demonstrate that the chosen directional geometric stiffness matrices retain only the bending modes of a dihedral element and eliminates other non-bending modes.

4.3.2 Orthotropic Geometric Stiffness. However, the difference between \( q_2 \) and \( q_3 \), for example, indicates that they are not reflection symmetric. If \( q_2 q_1^T \) and \( q_3 q_1^T \) are not imposed equally, coupling between \( t_1 \otimes m_1 \) and \( t_2 \otimes m_2 \) exists. To ensure reflection symmetry, some geometric stiffness pairs, such as \( q_2/q_3 \) and \( q_4/q_5 \), and \( q_6/q_7 \), should be applied equally, except for \( q_0/q_1 \) due to the presence of \( P \) in \( H \). Consequently, we define four reflection-symmetric geometric stiffness matrices, specifically,

\[
\begin{align*}
    S_0 &= (q_0 q_0^T + q_i q_i^T) / 2 = (t_1 t_1^T) \otimes (n_1 n_1^T) + (t_2 t_2^T) \otimes (n_2 n_2^T),
    S_1 &= (q_1 q_1^T + q_i q_i^T) / 2 = (t_1 t_1^T) \otimes (m_1 m_1^T) + (t_2 t_2^T) \otimes (m_2 m_2^T),
    S_2 &= (q_2 q_2^T + q_i q_i^T) / 2 = (t_1 t_1^T + t_2 t_2^T) \otimes (e e^T),
    S_3 &= (q_3 q_3^T + q_i q_i^T) / 2 = (s s^T) \otimes (n_1 n_1^T + n_2 n_2^T),
\end{align*}
\]

The resulting Hessian is

\[
H_q = \mu \left[ P P + \alpha_0 S_0 + \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 \right],
\]

where \( \alpha \) are independent geometric stiffness parameters. There exist orthogonality among \( \{S_0, S_1, S_2 \} \) due to the orthogonality among \( \{n_1, m_1, e\} \) and \( \{n_2, m_2, e\} \). Actually, they have significant implications:

\begin{itemize}
    \item \( S_0 \) damps the bending modes, i.e. the change of a dihedral angle \( \theta \), due to the effects of both \( q_0 q_1^T \) and \( q_1 q_1^T \).
    \item \( S_1 \) damps the altitude-changing modes. According to \( \nabla_x h_1 \) and \( \nabla_x h_2 \), we obtain an equivalent expression of \( S_1 \), i.e. \( S_1 = h_1^{-2} \nabla_x h_1 \nabla_x h_1^T + h_2^{-2} \nabla_x h_2 \nabla_x h_2^T \), which represents a stiffness matrix for altitude changes.
    \item \( S_2 \) damps the barycentric sliding modes, i.e. non-uniform scaling along \( e \), and \( S_1 \) damps undesirable scaling orthogonal to \( e \). \( \nabla_x o_1 \) and \( \nabla_x o_2 \) contain \( s \otimes m_1 \) and \( s \otimes m_2 \) which have influence on \( [h_1, h_2] \) and \( \theta \). If eliminating the effects of \( m_1 \) and \( m_2 \), we obtain an approximated expression of \( S_2 \), i.e. \( S_2 \approx \|e\|^2 \left( h_1^{-2} \nabla_x o_1 \nabla_x o_1 + h_2^{-2} \nabla_x o_2 \nabla_x o_2 \right) \), which represents a stiffness matrix for changes of barycentric weights.
\end{itemize}

4.3.3 Adaptive Parameters. However, determining the appropriate values for the intensity of these geometric stiffness can be challenging. Fortunately, we have successfully constructed an analytic eigensystem for the bending Hessian \( H \). We observe that the four geometric stiffness terms are already contained in \( Z Z^T \) and their coefficients can be extracted from the diagonal of \( P^T \). To be conservative, we choose the four parameters to be

\[
\begin{align*}
    \alpha_0 &= \max(F_{11}, F_{12}) - \beta, & \alpha_1 &= \max(F_{22}, F_{33}),
    \alpha_2 &= \max(F_{44}, F_{55}).
\end{align*}
\]

Since these parameters are adaptively determined based on eigenvalues, they will be zero when a dihedral element is at the reference state. By using F-based adaptive parameters, our AOGS not only enhances the stability of simulation of thin plates with high bending-membrane ratios, as shown in Fig. 7(d), but also produces little artificial damping for simulation with low bending-membrane ratios, as shown in Fig. 1(e, j) and Fig. 16(b).

4.4 Compression Stiffness Adjustment (CSA)

Our AOGS is effective in suppressing rapid over-stretching and over-compression caused by large bending forces in simulations with a high BMR, as illustrated in Fig. 10. However, our AOGS only modifies the energy Hessian and does not affect the bending force. After applying our geometric stiffness, compressed dihedral elements can be difficult to recover due to the large damping for in-plane deformation. This is because the magnitude of both bending stiffness and geometric stiffness increases rapidly and nonlinearly as the altitudes of a dihedral element decrease. While the membrane stiffness remains constant for a linear model, the bending stiffness is

\[
\text{ALGORITHM 2: Adaptive Orthotropic Geometric Stiffness}
\]

\[
\begin{align*}
\text{Input:} & \quad \text{Positions of a dihedral element: } \{x_0, x_1, x_2, x_3\} \\
\text{Output:} & \quad \text{Hessian with adaptive orthotropic geometric stiffness: } H_q \\
1 & \text{Compute projected } P'; \quad \triangleright \text{According to Algorithm 1} \\
2 & \text{Compute geometric stiffness: } S_0, S_1, S_2, S_3; \quad \triangleright \text{According to Eq. 13} \\
3 & \text{Compute parameters: } \alpha, \beta, \gamma, \xi; \quad \triangleright \text{According to Eq. 15} \\
4 & \text{Compute } H_q; \quad \triangleright \text{According to Eq. 14}
\end{align*}
\]

\[S_3 = (q_3 q_3^T + q_i q_i^T) / 2 = (s s^T \otimes (n_1 n_1^T + n_2 n_2^T)).\]
When using the Gauss-Newton approximation to fix the indefiniteness issue, we worsen the locking issue. High and low bending-membrane ratios are both common in interactive apparel design simulations, so a criterion must be established to invoke the nonlinear compression stiffness adjustment. In the study of thin materials, continuum materials typically adhere to the Kirchhoff-Love thin plate theory [Timoshenko et al. 1959], which states that the bending and membrane rigidity of a thin plate with Young’s modulus $Y$, Poisson’s ratio $\nu$, and thickness $h$ are denoted by

$$ D = \frac{h^3Y}{12(1-\nu^2)} \quad \text{and} \quad K = \frac{h^3Y}{1-\nu^2}, $$

(17)

respectively. Thus, the bending-membrane ratio of a Kirchhoff-Love thin plate is only related to its thickness, i.e. BMR: $D/K = h^4/12$. We use BMR x1: $1e^{-6}/12$ to represent a Kirchhoff-Love plate with a thickness of 1 mm. Given a BMR xN, the corresponding thickness is $\sqrt{N}$ mm. For example, BMR x16 corresponds to 4 mm.

In our experiments, as illustrated in Fig. 5 and Fig. 17, we have observed that H-projection, F-projection and the Gauss-Newton approximation are stable when simulating Kirchhoff-Love thin plates. Based on this observation, we suggest using the Kirchhoff-Love ratio as a criterion to determine when to apply the nonlinear compression stiffness adjustment. Using Eq. 17, we can calculate a compression stiffness $K_{KL} = 12h^{-2}D$ that satisfies the Kirchhoff-Love thin plate theory in terms of bending and compression. Once the linear compression stiffness falls below $K_{KL}$, the nonlinear compression stiffness adjustment will be invoked. To ensure that materials with extremely low compression stiffness receive enough resistance for compression quickly, we propose linearly interpolating the compression stiffness between $K_m$ and $K_{KL}$ when compression occurs. By combining this with the nonlinear expression in Eq. 16, we obtain the final expression for adjusting compression stiffness,

$$ \begin{cases} 
    f_m = K_m(r - 1), & \text{if } K_m \geq K_{KL}; \\
    f_m = \frac{K_m(r - 1)}{r^2}, & \text{if } K_m < K_{KL}.
\end{cases} $$

As depicted in Fig. 9, $f_m$ is also $C_1$ continuous at $r = 1$. As shown in Fig 10(e), we demonstrate the effectiveness of this approach in eliminating geometry degeneracy caused by over-compression. Using the Kirchhoff-Love ratio as a criterion provides an useful approach to balance stability and accuracy.

5 RESULTS

To demonstrate the effectiveness of our methods, we adopt a single Newton’s iteration with a large time step 1/30s, which is a commonly used strategy [Baraff and Witkin 1998] for implicit time integration. To solve linear systems to single-precision float-point accuracy, we use a preconditioned CG method with $3 \times 3$ block diagonal matrices for preconditioning. The PCG algorithm stops once the $L_2$ residual norm is below $1e^{-5}$. To eliminate nonlinearity arising from hyperelasticity, we adopt Baraff-Witkin’s linear elasticity model [Baraff and Witkin 1998] for in-plane deformation of unstructured triangle meshes, while our stabilized DAB models handle...
out-of-plane deformation. All of our experiments are performed on a PC with an Intel Core i9-12900K 3.20GHz CPU on 16 cores, 64GB of RAM, and an NVIDIA GeForce GTX 3090 GPU. We evaluate the effectiveness of our methods by challenging examples. As listed in Table 2, the material properties, including bending stiffness and membrane stiffness, of all examples are given.

5.1 Performance Evaluation

There are multiple methods to address the indefiniteness issue of DAB models, including the Gauss-Newton approximation, the Levenberg-Marquardt algorithm (LMA), the H-projection, and our proposed F-projection and AOGS. We have qualitatively assessed their stability by twisting and compressing cylinders in Fig. 5 and Fig. 7. Moreover, we quantitatively evaluate their performance in terms of solving linear systems and optimizing the nonlinear time-integration energy during the quasistatic process of a highly deformed armadillo (Fig. 13(b)) recovering to its reference state (Fig. 13(a)).

5.1.1 Convergence of solving linear systems. In the quasistatic simulation, a linear system needs to be solved at each Newton’s iteration. The strategy used to construct a positive semi-definite bending Hessian can affect the convergence of a block-diagonal preconditioned CG method for solving the linear system. Fig. 11 shows that our AOGS achieves the best convergence in terms of both relative residual and energy errors.

5.1.2 Convergence of optimizing nonlinear energy. When performing a quasistatic simulation of a heavily deformed armadillo and utilizing multiple Newton’s iterations to search for a nonlinear solution, a good bending Hessian strategy is crucial for achieving fast convergence with respect to nonlinear energy decrease. Fig. 12 demonstrates that our AOGS converges the fastest and always maintains a stable step length of one. Although LMA-0.5 outperforms our AOGS after 10 iterations, determining the optimal damping parameter of the LMA can be challenging.

5.1.3 Nonlinear Methods. While ensuring stability in nonlinear simulations can be achieved through small step lengths and line-searching methods, approaches that guarantee stability in linearly approximated simulations with only one Newton’s iteration are rare. Our proposed methods provide an effective solution for enhancing the stability of widely used DAB models in linear simulations and
more artificial damping as mesh resolution and bending stiffness increase. Nonetheless, our AOGS produces less artificial damping than the H-projection and F-projection, as demonstrated in our video. It is important to note that the Gauss-Newton approximation can result in underdamped results due to the absence of geometric stiffness. This can be observed in Fig. 16, where the pleated skirt simulated using the Gauss-Newton approximation exhibits dramatic swinging, while the result of our AOGS appears more plausible.

5.2 Artificial Damping

Andrews et al. [2017] highlighted in their work that inappropriate geometric stiffness can introduce artificial damping and energy dissipation into dynamic simulations. In our folding examples, as shown in Fig. 8, we observe the difference between results obtained using our AOGS and the Gauss-Newton approximation. To assess the side effects of artificial damping of our AOGS, we conducted a dynamic simulation of five plates with increasing resolutions from far to near as shown in Fig. 15. To reduce the influence of numerical damping from implicit time integration, we use a small time step 1/300 s in the simulation. We apply four different Hessian strategies to model bending deformation. When the bending stiffness is low, all four methods produce almost no artificial damping and are resolution-independent. However, only the Gauss-Newton approximation remains consistently resolution-independent as the bending stiffness increases. The other three methods produce increasingly

5.3 Bending-membrane Ratio

In the realm of thin materials, the dominant physical behaviors are in-plane membrane elasticity and out-of-plane bending deformations. This interplay is depicted in Fig. 1, where variations in the combination of membrane and bending properties yield distinct simulation outcomes for the same T-shirt. In practical scenarios, thin materials may exhibit either low membrane stiffness, rendering them prone to stretching, or high membrane stiffness, making them resistant to deformation. Similarly, these materials could possess low bending stiffness, allowing easy curvature, or high bending stiffness, resulting in curvature resistance.

The bending-membrane ratio (BMR) quantifies the relative interplay between these factors and mirrors the material thickness. Generally, most realistic thin materials exhibit a low BMR. As shown
in Fig. 17, existing methods can also produce stable simulation results. However, certain exceptional materials like specific scuba fabrics or composite materials can exhibit a high BMR. As shown in Fig. 8(a), there are oscillations on the folded scuba T-shirt. Importantly, DAB models are decoupled from membrane constitutive models. For user friendliness and convenience, we empower users to independently fine-tune bending and membrane parameters without a deep understanding of Kirchhoff-Love theory or real-world material parameters. Consequently, it becomes straightforward to create some non-physical material parameters, even those with high BMRs. Even so, maintaining simulation stability without compromising the interactive experience remains paramount.

Illustrated in Fig. 7, 8, 10, we have showcased the efficacy of our proposed orthotropic geometric stiffness (AOGS) and compression stiffness adjustment (CSA) in stabilizing simulation of thin materials using only one Newton’s iteration with PCG as a linear solver. In combination with a linear membrane model, such as the Baraff-Witkin’s model, our method facilitates the progression of Newton’s iterations without the computational overhead of expensive line searches. This integration ensures the maintenance of interactive fluidity and responsiveness. Nonetheless, we must acknowledge that the advantages conferred by our approach in scenarios featuring a low BMR might not be as pronounced as those in high BMR cases. In the examples involving a crushing coke can in Fig. 18, whether incorporating plasticity or not, we note that the Gauss-Newton approximation can similarly facilitate Newton’s iterations without line searches. This outcome arises from the fact that adequate membrane stiffness can mitigate the adverse consequences stemming from the absence of significant bending geometric stiffness.

5.4 Different DAB Models
By default, we use Discrete Shells as the DAB model in previous examples to simulate bending deformation. However, our AOGS is not limited to Discrete Shells, as we have also applied it to other DAB models to improve their stability, including Cubic Shells, Bridson’s model and Tamstorf’s model. In the case of large bending deformation, simply omitting the indefinite dihedral angle Hessian $G$ is inadequate for Cubic Shells and Tamstorf’s model to ensure positive semi-definiteness because $p$ in Eq. 2 can be negative. Additionally, Tamstorf’s model is more unstable than others due to its unbounded bending energy. As shown in Fig. 19, our AOGS can enhance stability of different DAB models, allowing a highly deformed armadillo with many (near-) degenerate geometries to stably recover to its reference state.

6 CONCLUSIONS, LIMITATIONS AND FUTURE WORK
This paper presents a concise matrix expression for the energy Hessian of dihedral angle based discrete bending models and proposes an innovative analytic expressions for the eigen system of DAB models to address the indefiniteness issue. Based on the twelve motion modes of a dihedral element, we demonstrate that original geometric stiffness matrix of DAB models retains the undesirable altitude-changing modes and propose an orthotropic geometric stiffness model with adaptive parameters which retains the desirable bending modes and can improve simulation stability at degenerate geometries. Our method supports stable simulations of thin materials with a wide range of bending-membrane ratios. Nonetheless,
artificial damping is noticeable when simulating high-resolution meshes with high bending stiffness. In addition, our compression stiffness adjustment inherits the property of Kirchhoff-Love thin plate that introduces some coupling between in-plane compression and out-of-plane bending. To further improve bending models for discrete surfaces, future work could focus on eliminating the influence of mesh tessellation on edge-based discrete bending models and addressing the well-known locking issue when the bending-membrane ratio is low.

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We summarize them as:

When a dihedral element has bending deformation, i.e. the dihedral angles, we would like to reformulate it in matrix format so that we can do not present them here in detail. Tamstorf and Grinspun [2013] provide derivations of gradients of the other five variables are simple so we should not consider them further. Our discovery of the analytic expression for the eigensystem of the energy Hessian of DAB models involves three key steps: the discovery of matrix $Z$, the discovery of the intrinsic bending matrix $F$, and the analytic eigensystem of $F$.

B ANALYTIC EIGENSYSTEM

At the beginning of analyzing the eigensystem of the energy Hessian, we face considerable challenges and uncertainties. However, through a careful examination of the twelve motion modes described by Bridson et al. [2003], we discover that eight vectors, as presented in Eq. 23, we get the final formula for the symmetrical dihedral angle Hessian in Eq. 4.

\begin{align}
\nabla \mathbf{n}_1 &= t_1 \otimes (m_1^T) + s \otimes (n_1 e^T), \\
\nabla \mathbf{n}_2 &= t_2 \otimes (m_2^T) + s \otimes (n_2 e^T).
\end{align}

Substituting Eq. 21 and Eq. 22 into Eq. 20 getting us

\begin{align}
\nabla (t_1 \otimes \mathbf{n}_1) &= (t_1)_{12} \mathbf{A}_1 + (B_1 + B_2) + (s)_{12} \mathbf{m}_1, \\
\nabla (t_2 \otimes \mathbf{n}_2) &= (t_2)_{12} \mathbf{A}_2 + (B_2 + B_2) + (s)_{12} \mathbf{m}_2.
\end{align}

where $A_1 = m_1^T + n_1 m_1^T$ and $A_2 = m_2^T + n_2 m_2^T \in \mathbb{R}^{3x3}$, $B_1 = (t_1 s) \otimes (e^T)$ and $B_2 = (t_2 s) \otimes (e^T) \in \mathbb{R}^{12x12}$. Let $C = m_1^T + m_2^T \in \mathbb{R}^{3x3}$, we can get $A_1 + A_2 = C + C^T$. $C$ have been proved symmetrical by Tamstorf and Grinspun [2013]. Therefore, we have $C = (A_1 + A_2)/2$. By reformulating Eq. 23, we get the final form of the analytic expression of the dihedral angle.

\begin{align}
\nabla \mathbf{h}_1 &= h_{11}^{-1} (s \otimes \mathbf{n}_1 + t_1 \otimes (e^T)), \\
\nabla \mathbf{h}_2 &= h_{12}^{-1} (s \otimes \mathbf{n}_2 + t_2 \otimes (e^T)).
\end{align}

In matrix format, we get $G = \nabla (t_1 \otimes \mathbf{n}_1) + \nabla (t_2 \otimes \mathbf{n}_2)$, in which

\begin{align}
\nabla (t_1 \otimes \mathbf{n}_1) &= (t_1)_{12} \mathbf{A}_1 + (B_1 + B_2) + (s)_{12} \mathbf{m}_1, \\
\nabla (t_2 \otimes \mathbf{n}_2) &= (t_2)_{12} \mathbf{A}_2 + (B_2 + B_2) + (s)_{12} \mathbf{m}_2.
\end{align}

Therefore, the gradients of $t_1, t_2, n_1$ and $n_2$ are needed. In isometric simulation, $t_1$ and $t_2$ are constant. However, if they are variant during non-isometric simulation, their derivatives

\begin{align}
\nabla \mathbf{h}_1 &= h_{11}^{-1} (s \otimes \mathbf{h}_1), \\
\nabla \mathbf{h}_2 &= h_{12}^{-1} (s \otimes \mathbf{n}_2 - t_2 \otimes \mathbf{h}_2).
\end{align}

should be taken into consideration. The gradient of triangle normals are borrowed from Tamstorf and Grinspun [2013], which are

\begin{align}
\nabla \mathbf{n}_1 &= t_1 \otimes (m_1^T) + s \otimes (n_1 e^T), \\
\nabla \mathbf{n}_2 &= t_2 \otimes (m_2^T) + s \otimes (n_2 e^T).
\end{align}

\[Z = [\xi_0, \ldots, \xi_8] \in \mathbb{R}^{12x8} \] is composed of the eight $\xi$s in Eq. 6.
B.2 Intrinsic Bending Deformation

Furthermore, we discover $HZ = ZC$ where $C \in \mathbb{R}^{8 \times 8}$ is an important coefficient matrix. This revelation indicates that the column space of $Z$ is an invariant subspace of $H$ and $Z$ is closely related to the corresponding eigenvectors, denoted as $E = \{e_0, \ldots, e_{11}\}$.

For any arbitrary vector $y \in \mathbb{R}^{12}$, it can be easily verified that $e = Hy = Zy$ where $y \in \mathbb{R}^{3}$ is a coefficient vector. Consequently, we can determine that $e$ represents the general form of eigenvectors corresponding to nonzero eigenvalues. To ensure orthogonality between two different eigenvectors $e_i$ and $e_j$, we require $e_i^T e_j = y_i^T Wy_j = 0$, where $W = Z^T Z$ is positive-definite due to the full column rank of $Z$. Since $e$ is an eigenvector, we have $HZy = \lambda Zy$, which leads to $ZCy = \lambda Zy$. Consequently, we can conclude that $Cy = \lambda y$, implying that the eigenvalues of $C$ are also eigenvalues of $H$ and $y$ is the eigenvector of $C$. Hence, the orthogonality between eigenvectors gets us the following two conditions,

$$
\begin{align*}
(y_i^T Wy_j = 0 & \quad \text{if } i \neq j, \\
(y_i^T C y_j = 0 & \quad \text{if } i \neq j. \tag{24}
\end{align*}
$$

Actually, there exists a close connection between $C$ and $W$, specifically $C = FW$, where $F$ is a symmetric block diagonal matrix that is solely related to the current and reference dihedral angle, $\theta$ and $\tilde{\theta}$. The detailed deduction process of $F$ is tedious and can be found in the subsequent section, B.3. Fundamentally, $F$ represents the bending deformation of a dihedral element, while $W$ represents the in-plane information of a dihedral element.

We have presented the analytic eigensystem of DAB models in our paper. A significant finding is that $F$ typically has positive, negative and zero eigenvalues, with four of each, when a dihedral element is apart from its reference bending state. However, the eigensystem of $F$ differs from the eigensystem of $H$ due to the lack of orthogonality among the columns of $Z$. Despite our attempts to derive the exact analytic eigensystem of $H$ by utilizing Eq. 24, we were unsuccessful. Nevertheless, we have proven that $H$ also typically has four positive and four negative eigenvalues, as well as four zero eigenvalues.

**Proof.** According to Eq. 7 and Eq. 10, we obtain a useful decomposition for the energy Hessian, namely $H = \mu(ZE)A(ZE)^T$, in which $ZE \in \mathbb{R}^{12 \times 8}$ has full column rank. If the matrix $ZE$ had orthogonal column vectors, we could easily conclude that $H$ would have four positive and four negative eigenvalues, as $A$ has four positive and four negative eigenvalues. However, $ZE$ does not have orthogonal columns. By performing QR decomposition to $ZE$, we obtain an matrix $Q \in \mathbb{R}^{12 \times 8}$ with orthogonal unit column vectors, i.e. $ZE = QR$ where $R \in \mathbb{R}^{8 \times 8}$ is a upper-triangular matrix with full rank. Consequently, we obtain another equivalent decomposition for the energy Hessian, i.e. $H = \mu Q T Q^T$, in which the matrix $T = R A R^T$ has the same number of negative eigenvalues as $A$, and the same applies to the number of positive eigenvalues. This results follows Sylvester’s law of inertia, provided that $R$ is an invertible matrix. Thus, we conclude that $H$ also have the same numbers of negative eigenvalues and positive eigenvalues as $A$. \qed

B.3 The discovery of $F$

Actually, the derivation of $F$ is a nontrivial task. Specifically, we carefully deduce the following eight expressions that can be used to construct matrix $F$. These expressions ultimately lead us to the important result, $HZ = \mu Z F Z^T$. Due to the full column rank of $Z$, we can further derive Eq. 7 and obtain the analytic expression for matrix $F$.

$$
\begin{align*}
H_{\xi_0} &= \mu \left[ t_1^T t_1 \cos \theta + (t_2^T t_2) \right] \left[ t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) \right] + \\
&\mu \left[ (t_1^T t_1) t_1 \otimes m_1 + (t_2^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T t_1 \right] s \otimes \left[ -\frac{1}{2} \sin 2\theta n_1 + \sin^2 \theta m_1 + (t_1 - \cos \theta t_2) \otimes e \right], \\
H_{\xi_1} &= \mu \left[ t_1^T t_2 \cos \theta + (t_2^T t_2) \right] \left[ t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) \right] + \\
&\mu \left[ (t_1^T t_2) t_1 \otimes m_1 + (t_2^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T t_2 \right] s \otimes \left[ -\frac{1}{2} \sin 2\theta n_1 + \sin^2 \theta m_1 + (t_1 - \cos \theta t_2) \otimes e \right], \\
H_{\xi_2} &= \mu \left[ t_2^T s \cos \theta + (s^T t_2) \right] \left[ t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) \right] + \\
&\mu \left[ (t_2^T s) t_1 \otimes m_1 + (s^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T s \right] s \otimes \left[ -\frac{1}{2} \sin 2\theta n_1 + \sin^2 \theta m_1 + (t_1 - \cos \theta t_2) \otimes e \right], \\
H_{\xi_3} &= \mu \left[ t_1^T t_1 \right] t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) + \\
&\mu \left[ (t_1^T t_1) t_1 \otimes m_1 + (t_2^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T t_1 \right] s \otimes \left[ (\sin \theta n_1 + \cos \theta m_1) + t_2 \otimes e \right], \\
H_{\xi_4} &= \mu \left[ t_1^T t_1 \right] t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) + \\
&\mu \left[ (t_1^T t_1) t_1 \otimes m_1 + (t_2^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T t_2 \right] s \otimes \left[ (\sin \theta n_1 + \cos \theta m_1) + t_2 \otimes e \right], \\
H_{\xi_5} &= \mu \left[ t_2^T t_2 \right] t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) + \\
&\mu \left[ (t_2^T t_2) t_1 \otimes m_1 + (s^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T s \right] s \otimes \left[ (\sin \theta n_1 + \cos \theta m_1) + t_2 \otimes e \right], \\
H_{\xi_6} &= \mu \left[ t_1^T t_1 \right] t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) + \\
&\mu \left[ (t_1^T t_1) t_1 \otimes m_1 + (t_2^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T t_1 \right] s \otimes \left[ (\sin \theta n_1 + \cos \theta m_1) + t_2 \otimes e \right], \\
H_{\xi_7} &= \mu \left[ t_2^T t_2 \right] t_1 \otimes n_1 + t_2 \otimes (\cos \theta n_1 + \sin \theta m_1) + \\
&\mu \left[ (t_2^T t_2) t_1 \otimes m_1 + (s^T t_2) t_2 \otimes (\sin 2\theta n_1 - \cos 2\theta m_1) \right] + \\
&\mu \left[ s^T s \right] s \otimes \left[ (\sin \theta n_1 + \cos \theta m_1) + t_2 \otimes e \right].
\end{align*}
$$